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# Asymptotic Solutions for High-Frequency Trapped Wave Propagation

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# ASYMPTOTIC SOLUTIONS FOR HIGH-FREQUENCY TRAPPED WAVE PROPAGATION

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CONTENTS



A technique is developed by which high-frequency trapped scalar wave problems are reduced to finding solutions for a sequence of equations which are independent of the frequency. The technique is applied to whispering gallery waves, ducted waves, edge waves and surface waves. Several successful comparisons are made between the asymptotic expansions of exact analytic solutions and the results obtained by using the technique while one numerical comparison shows that such results can be extremely accurate even when the frequency is low. The range of validity of the solutions is discussed with particular reference to numerical solutions.

## 1. Introduction and outline of the technique

### 1.1. *Introduction*

There are many physical situations where waves have large amplitudes only near certain lines or surfaces. For example gravity surface waves on deep water become insignificant several wavelengths below the surface which explains why submarines can travel smoothly even during

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severe storms. Because their energy is located within a small distance of a line or surface 'trapped' waves are particularly efficient in guiding energy along that surface for either useful or destructive purposes. Consequently trapped waves are of practical as well as theoretical interest.

Some exact analytic solutions of linear reduced wave equations are known for several simple classes of trapped waves in highly idealized situations. For a real situation such solutions often only give a qualitative guide to the wave's behaviour. In principle it is possible to solve a linear reduced wave equation directly by using numerical methods, however, when the frequency of the waves is high the wavelength is small so any numerical process would require a very fine network of calculation points in order to converge, in addition the computation would have to be repeated for each frequency present.

An analogous situation is encountered in seeking solutions of the reduced scalar wave equation

$$\nabla^2 u + k^2 N^2 u = 0, \tag{1.1.1}$$

which correspond to scalar waves $u(\boldsymbol{x})$ of frequency $k$ which are propagating in a non-homogeneous medium of refractive index $N$ far away from such complications as boundaries or caustics. For this problem there is the classical geometric optics' solution, where in order to retain the real geometry we no longer demand an exact analytic solution but accept instead a solution which is asymptotically correct as $k \to \infty$. The solution is given by

$$u \sim A \exp(\mathrm{i}k\phi), \tag{1.1.2}$$

provided that $A$ and $\phi$ satisfy the two equations

$$(\nabla\phi)^2 = N^2, \tag{1.1.3}$$

and

$$2\nabla\phi \, . \, \nabla A + A\nabla^2\phi = 0. \tag{1.1.4}$$

These two equations, the 'eikonal' and 'transport' equations of geometrical optics, are independent of $k$ and can be solved simply by marching along the directions $\nabla\phi$ since along these ray paths the two equations become ordinary differential equations. Once $A$ and $\phi$ are known it is a trivial matter to reconstruct an asymptotic approximation for $u$ at a given frequency.

An extension of the geometrical optics' solution (1.1.2) has been applied to trapped waves by Grimshaw (1968). He shows how permitting $\phi$ to be complex and replacing $A$ by an expansion in inverse powers of $k$ enables us to find solutions which are accurate to any given inverse power of $k$ for a certain class of surface wave problems. This success might suggest that by similar direct extensions of the geometric optics' solution (1.1.2) all the other classes of trapped waves could be solved. Unfortunately this is not the case and, as will be shown in § (5.5), Grimshaw's solution has several undesirable features which are not present in solutions obtained when the technique described here for solving scalar trapped wave problems is applied to surface waves.

Although not a direct extension of the geometrical optics' solution the new technique has the same major simplifying feature namely that the final equations to be solved are all independent of the frequency. The next subsection gives a brief outline of the general features of this technique prior to the more complete treatment given in §§ 2 to 5 when the technique is applied to whispering gallery waves, ducted waves, edge waves and surface waves.

### 1.2. *Outline of the technique*

A trapped wave appears to be a local phenomenon and therefore we should expect to be able to find asymptotic solutions for trapped waves even if we had no knowledge about conditions far away from the line or surface $S$ to which the wave is trapped. Unfortunately the very concept of a trapped wave implies a boundary condition far away from $S$ namely that the amplitude tends to zero. We shall escape this dilemma by specifying the functional form of the wave; a step which introduces several new difficulties.

The asymptotic solution must give the correct limiting behaviour very close to $S$ as the frequency approaches infinity, correspondingly the first step in the technique is to find this 'fundamental solution' for that class of trapped waves

$$u = f_1(k^l r)\exp(\mathrm{i}k^m s), \tag{1.2.1}$$

in which $k$ is proportional to the frequency, $l$ and $m$ are rational numbers, $r$ measures along wavefronts, $s$ measures perpendicular to wavefronts and $f_1(z)$ satisfies a differential equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(p\frac{\mathrm{d}f_1}{\mathrm{d}z}\right) + qf_1(z) = 0, \tag{1.2.2}$$

where $p$ and $q$ are functions of $z$. For example in §§ 2.2, 3.2, 4.2 and 5.2 $f_1$ is found to be an Airy function, a Hermite function, a Laguerre function and an exponential function.

Starting from this fundamental solution it can be deduced that in order to satisfy the scalar reduced wave equation near $S$ when $k$ is large but not infinite the general solution should have the functional form

$$u = \exp(\mathrm{i}k^m\sigma)\{Af_1(k^l\rho) + \mathrm{i}k^n Bf_2(k^l\rho)\}, \tag{1.2.3}$$

where $A$, $B$, $\rho$ and $\sigma$ are slowly varying functions of position which may depend upon $k$, $n$ is a rational number and $f_2(z)$ is defined by

$$f_2(z) = p(z)\,\mathrm{d}f_1(z)/\mathrm{d}z. \tag{1.2.4}$$

If we assume that $\rho$ and $\sigma$ are independent of $k$ we can find asymptotic solutions for certain simple problems including waves across a fixed isolated caustic (Ludwig 1966), waves with two caustics near each other (§ 3.4), edge waves in the absence of the Coriolis force (§ 4.4) and a class of surface wave problems (Grimshaw 1968). In all of these problems we find that $A$ and $B$ must be complex.

It seems more physically meaningful to assume that $A$, $B$, $\rho$ and $\sigma$ are all real and to interpret them as the amplitude, an amplitude correction due to the curvature of $S$, the decay measurement and the phase measurement. Using these new assumptions we can obtain asymptotic solutions for all the above problems and many other problems which cannot be solved by a technique based upon the previous assumptions.

When the general solution (1.2.3) is substituted into the reduced wave equation by using the relationships (1.2.2) and (1.2.4) we can obtain an expression of the form

$$C(\boldsymbol{x}, k)f_1(k^l\rho) + D(\boldsymbol{x}, k)f_2(k^l\rho) = 0. \tag{1.2.5}$$

From this equation we first determine the value of $n$ by demanding that as $k \to \infty$ the $C$ and $D$ terms are of the same order. It is then necessary that both $C$ and $D$ are zero since in general $f_1$ and $f_2$ are independent functions whose ratio can vary rapidly with position while $C/D$ being a function of $A$, $B$, $\rho$ and $\sigma$ can only vary slowly. Also since $A$, $B$, $\rho$ and $\sigma$ are all real the real and imaginary parts of $C$ and $D$ must be zero separately. Thus we now have four equations satisfied by the four unknowns $A$, $B$, $\rho$ and $\sigma$.

The boundary conditions on $S$ present far more difficulties than the reduced wave equation. For example the above argument about the independence of $f_1$ and $f_2$ cannot be used since $\rho$

cannot vary much on $S$. We can of course obtain two boundary conditions by taking real and imaginary terms separately. In several circumstances additional boundary conditions have to be found by *ad hoc* physical or mathematical arguments (§§ 4.3 and 5.3).

At this stage we utilize the fact that we only require solutions which are asymptotically correct as $k \to \infty$ to expand $A$, $B$, $\rho$ and $\sigma$ as asymptotic series

$$A \sim A_0 + A_1/k^g + \ldots, \tag{1.2.6}$$

$$B \sim B_0 + B_1/k^g + \ldots, \tag{1.2.7}$$

$$\rho \sim r + r_1/k^g + \ldots \tag{1.2.8}$$

and

$$\sigma \sim s + s_1/k^g + \ldots \tag{1.2.9}$$

in which $A_j$, $B_j$, $r_j$ and $s_j$ are all independent of $k$ and the necessary value of $g$ is usually quite clear from the equations and boundary conditions satisfied by $A$, $B$, $\rho$ and $\sigma$. We finally substitute these expansions into the equations and boundary conditions and extract like powers of $k$ to give an infinite sequence of equations and boundary conditions all of which are independent of $k$.

The analytical processing of the problem is now completed and like the eikonal and transport equations of geometrical optics the resulting sequence of equations are all considerably simpler to solve than the original problem whether by analytic or by numerical methods in all of the applications examined in this paper. Also once the first few of these equations have been solved it is a trivial matter to reconstruct an asymptotic approximation for $u$ at a given frequency.

It is only when we attempt to solve this sequence of equations that the distinction between problems in which $S$ is a line and problems in which $S$ is a surface becomes apparent. If $S$ is a line trapped waves can either travel up or down $S$ and there are at most a denumerable number of modes present all of which have the $r$ and $s$ coordinates in common. However, if $S$ is a surface, trapped waves can propagate in any direction across $S$ so in general there will be a continuum of independent waves. Consequently when $S$ is a surface we are restricted to problems in which there is a preferred direction across $S$, for example trapped waves radiating outwards from a curve $C$ on $S$ along which the contribution from each wave mode to $u$ is known.

In order to support the contention that the technique provides a practical (as opposed to a formal) method for solving trapped wave problems, the applications have been chosen primarily because of their physical interest. Correspondingly a certain amount of physical background is given for each of the applications to show why the application is of interest and also to keep this paper self-contained. Likewise the technique is related to the more physical ray explanations of the qualitative behaviour of trapped waves and whenever the alternative technique introduced above can be applied, the two techniques are compared.

Although the technique is envisaged as being used in conjunction with numerical procedures in order to obtain accurate quantitative results in real situations, it can also be used to give results of a more theoretical kind. For example in § 3.5 it is noted that neither the amplitude nor the phase-velocity of long waves trapped over an oceanic ridge are sensitive to the Coriolis force. Again in § 4.5 it is noted that the leading term in the increase of water level at the shore is inversely proportional to the beach slope. This second result could serve as a guide to how high flood-barriers should be at points for which detailed water-level records are not known.

## 2. Whispering gallery waves

### 2.1. *Introduction*

St Paul's Cathedral is renowned for its remarkable Whispering Gallery, situated high up inside the building around the base of its vast dome. In this gallery you can often hear sounds surprisingly far away from their origin when they are both made and heard close to the wall. It is very noticeable how high frequencies are heard most easily; for example, a whisper carries further than normal speech, the higher frequency more than compensating for the reduced energy, this observation over many years led to the name of the gallery.

The accepted mechanism for the phenomenon was first explained by Lord Rayleigh (1878). If we regard sound as composed of rays diverging from their source, the whispering gallery phenomenon is due to those rays which are repeatedly reflected by the wall as shown in figure 1. The other rays are subject to the usual inverse square law for wave propagation in three dimen-

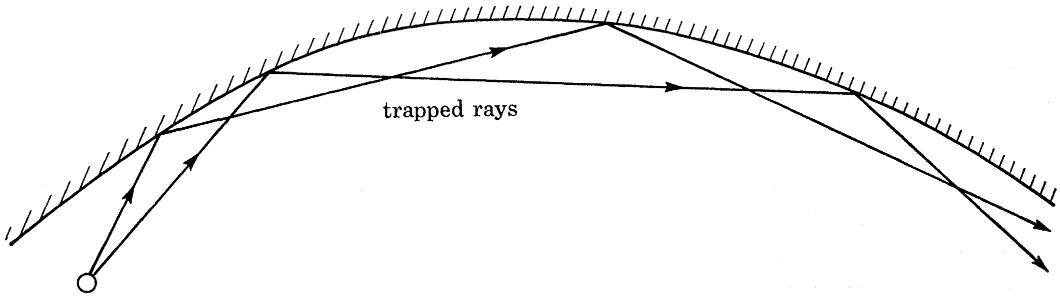


Figure 1

sions, however the multiply-reflected whispering gallery rays being trapped near the surface are propagating in only two dimensions so diminish in intensity by the inverse first power law. This less rapid enfeeblement of sound propagating in the whispering gallery modes is one of the main reasons for the surprisingly large distances at which sounds can be heard in the gallery.

Several years later, 1910, Lord Rayleigh solved a special problem of whispering gallery waves propagating around a circular cylinder. He showed, by approximating his solution when the frequency was high, that, for the simplest mode, the ratio of the effective width of the sound from the wall to the radius of curvature of the wall varies inversely as the two thirds power of the frequency. In the light of later advances in the study of Bessel's functions we can readily verify the same result for any whispering gallery mode. This explains the second feature of whispering gallery waves; high-frequency sounds are more intense than low-frequency sounds because their energy is packed into a narrow zone near to the wall and not spread out thinly.

Once the particular problem in acoustics was understood it became clear that in any other situations where waves propagate similar phenomena can occur. Lord Rayleigh suggested that one example might be earthquake waves, and indeed Love waves are of whispering gallery type, though the repeated reflexion of rays is mainly due to the inhomogeneity of the Earth and not, as Rayleigh suggested, due to the curvature of the Earth's surface. Another example is the propagation of long radio waves far beyond the optical horizon by multiple reflexions at the ionosphere which acts like the wall of a whispering gallery (Budden & Martin 1962).

Although the theoretical aspects of whispering gallery waves, the mechanism and the qualitative behaviour, are now well known there still remains the practical problem of predicting accurately the results in real situations. Out of all the many different fields in which whispering gallery phenomena are found we shall, in this section, restrict our attention to the problem of scalar waves $u(\mathbf{x})$ which satisfy the reduced wave equation

$$\nabla^2 u + k^2 N^2 u = 0 \tag{2.1.1}$$

and are subject to the linear homogeneous boundary condition

$$\partial u/\partial n + k^{\frac{2}{3}}\zeta u = 0 \tag{2.1.2}$$

on the surface $S$. Where $N$ is the refractive index of the medium, $\partial/\partial n$ denotes differentiation along the normal to the surface $S$ and $\zeta$ is a real function of position on $S$.

The possibly unfamiliar two-thirds power in the boundary condition has been chosen so that the two most common boundary conditions $u = 0$ or $\partial u/\partial n = 0$ can be dealt with in a single calculation simply by setting $\zeta = \infty$ or $\zeta = 0$.

### 2.2. *Fundamental solution for whispering gallery waves*

As the frequency increases the effective width of a fixed whispering gallery wave mode shrinks as $k^{-\frac{2}{3}}$, on this scale the properties of the medium or boundary vary very slowly. The curvature is of course significant in determining the existence and properties of whispering gallery waves as well as the inhomogeneity of the refractive index. We can, however, ignore curvature by using the well-known artifice of a modified refractive index, i.e. an observer who thought that the boundary was flat would attribute the bending of a ray back to the boundary as being entirely due to the inhomogeneity of the refractive index. Thus in order to find the 'fundamental solution' for whispering gallery waves we need only study those mathematically more convenient problems in which the boundary $S$ is perfectly flat, the refractive index $N$ is a function only of the perpendicular distance $x$ from the boundary and $\zeta$ is a real constant.

We seek solutions of equations (2.1.1) and (2.1.2) which correspond to whispering gallery waves propagating along $S$:

$$u = \hat{u}(x)\exp(ikly). \tag{2.2.1}$$

By substituting this representation into equations (2.1.1) and (2.1.2) together with the trapped wave requirement that $u$ tends to zero far away from $S$, we get an eigenvalue problem for $\hat{u}$ and $l$:

$$\mathrm{d}^2\hat{u}/\mathrm{d}x^2 + k^2(N^2 - l^2)\,\hat{u} = 0, \tag{2.2.2}$$

$$\text{on } x = 0 \quad \mathrm{d}\hat{u}/\mathrm{d}x + k^{\frac{2}{3}}\zeta\hat{u} = 0, \tag{2.2.3}$$

and as $x \to \infty$

$$\hat{u} \to 0. \tag{2.2.4}$$

It is of course essential that $N$ decreases outwards from $S$.

The local form of solutions near $x = 0$ as $k \to \infty$ is easily obtained by first making the transformations

$$\xi = k^{\frac{2}{3}}x \tag{2.2.5}$$

and

$$l = N(0) + k^{-\frac{2}{3}}\hat{\ell}, \tag{2.2.6}$$

and then letting $k \to \infty$. A particularly simple eigenvalue problem is found for $\hat{u}_\infty(\xi)$ and $\hat{\ell}_\infty$ with the solutions

$$\hat{u}_\infty = C\,\mathrm{Ai}(\alpha\xi + \beta), \tag{2.2.7}$$

$$l_\infty = \tfrac{1}{2}\beta\alpha^2/N(0), \tag{2.2.8}$$

where $$\alpha = \left(-2N(0)\frac{dN}{dx}\bigg|_0\right)^{\frac{1}{3}}, \tag{2.2.9}$$

$$\alpha\,\mathrm{Ai}'(\beta)+\zeta\,\mathrm{Ai}(\beta) = 0, \tag{2.2.10}$$

$C$ is an arbitrary constant and Ai is an Airy function.

This local solution defines the 'fundamental solution' for whispering gallery waves. In the notation of § 1.2

$$f_1(t) = \mathrm{Ai}(t) \tag{2.2.11}$$

and $f_1$ satisfies the ordinary differential equation

$$d^2f_1/dt^2 - tf_1 = 0, \tag{2.2.12}$$

therefore $$f_2(t) = \mathrm{Ai}'(t). \tag{2.2.13}$$

### 2.3. *Derivation of equations*

We now return to the general situation and seek solutions of equations (2.1.1) and (2.1.2) which correspond to whispering gallery waves propagating along a smooth curved surface $S$ bounding on non-homogeneous medium and on which $\zeta$ is a function of position. As was indicated § 1.2 we use the 'fundamental solution' to obtain the general solution for whispering gallery waves:

$$u = \exp(ik\sigma)\,\{A\,\mathrm{Ai}(k^{\frac{2}{3}}\rho)+ik^{-\frac{1}{3}}B\,\mathrm{Ai}'(k^{\frac{2}{3}}\rho)\}, \tag{2.3.1}$$

where $A$, $B$, $\rho$, $\sigma$ are all real slowly varying functions which may depend on $k$ and the $k^{-\frac{1}{3}}$ factor in the second term is found by the requirement that when $u$ is substituted into equation (2.1.1) the $A$ and $B$ terms are of the same order in $k$.

By substituting the general solution (2.3.1) directly into the reduced wave equation (2.1.1), using Airy's equation (2.2.12) to replace Ai″ in terms of Ai, we get

$$\begin{aligned} &k^2\,\mathrm{Ai}\{A[\rho(\nabla\rho)^2-(\nabla\sigma)^2+N^2]-2B\rho\,\nabla\rho\,.\,\nabla\sigma\} \\ &\quad +k^{\frac{5}{3}}\mathrm{i}\,\mathrm{Ai}'\{2A\,\nabla\rho\,.\,\nabla\sigma+B\rho(\nabla\rho)^2-(\nabla\sigma)^2+N^2\} \\ &\quad +k\mathrm{i}\,\mathrm{Ai}\{\nabla\,.\,(A\nabla\sigma)+\nabla\sigma\,.\,\nabla A+\nabla\,.\,(B\rho\,\nabla\rho)+\rho\,\nabla\rho\,.\,\nabla B\} \\ &\quad +k^{\frac{2}{3}}\,\mathrm{Ai}'\{\nabla\,.\,(A\nabla\rho)+\nabla\rho\,.\,\nabla A-\nabla\,.\,(B\,\nabla\sigma)-\nabla\sigma\,.\,\nabla B\} \\ &\quad +\mathrm{Ai}\{\nabla^2A\}+k^{-\frac{1}{3}}\mathrm{i}\,\mathrm{Ai}'\{\nabla^2B\} = 0. \end{aligned} \tag{2.3.2}$$

Following the general principles expounded in § 1.2 we must put the coefficients of Ai and Ai′ separately to zero and then because we have assumed $A$, $B$, $\rho$, $\sigma$ are all real we take the real and imaginary parts giving in all four equations. After some algebraic manipulation we obtain the following set of equations

$$\rho(\nabla\rho)^2-(\nabla\sigma)^2+N^2+k^{-2}\frac{A\nabla^2A+\rho B\nabla^2B}{A^2+\rho B^2} = 0, \tag{2.3.3}$$

$$\nabla\rho\,.\,\nabla\sigma+\tfrac{1}{2}k^{-2}\frac{A\nabla^2B-B\nabla^2A}{A^2+\rho B^2} = 0, \tag{2.3.4}$$

$$\nabla\,.\,(A\nabla\sigma)+\nabla\sigma\,.\,\nabla A+\nabla\,.\,(B\rho\nabla\rho)+\rho\nabla\rho\,.\,\nabla B = 0 \tag{2.3.5}$$

and $$\nabla\,.\,(A\nabla\rho)+\nabla\rho\,.\,\nabla A-\nabla\,.\,(B\nabla\sigma)-\nabla\sigma\,.\,\nabla B = 0. \tag{2.3.6}$$

Similarly, by substituting the general solution (2.3.1) into the boundary condition (2.1.2), using Airy's equation (2.2.12) to replace Ai″ in terms of Ai, separating real and imaginary parts we obtain the two conditions

$$\mathrm{Ai}(k^{\frac{2}{3}}\rho)\left[A\frac{\partial \sigma}{\partial n}+B\rho\frac{\partial \rho}{\partial n}\right]+k^{-\frac{2}{3}}\,\mathrm{Ai}'(k^{\frac{2}{3}}\rho)\,B\zeta+k^{-\frac{4}{3}}\,\mathrm{Ai}'(k^{\frac{2}{3}}\rho)\frac{\partial B}{\partial n}=0 \tag{2.3.7}$$

and

$$A\left[\mathrm{Ai}'(k^{\frac{2}{3}}\rho)\frac{\partial \rho}{\partial n}+\zeta\,\mathrm{Ai}(k^{\frac{2}{3}}\rho)\right]+\mathrm{Ai}'(k^{\frac{2}{3}}\rho)\,B\frac{\partial \sigma}{\partial n}+k^{-\frac{2}{3}}\,\mathrm{Ai}(k^{\frac{2}{3}}\rho)\frac{\partial A}{\partial n}=0 \tag{2.3.8}$$

to be satisfied on the boundary $S$.

From the equations (2.3.3) to (2.3.6) and most significantly the boundary conditions (2.3.7) and (2.3.8) we deduce that as $k\to\infty$ $A$, $B$, $\rho$ and $\sigma$ must possess the following asymptotic expansions

$$A \sim A_0+A_1/k^{\frac{2}{3}}+\ldots, \tag{2.3.9}$$

$$B \sim B_0+B_1/k^{\frac{2}{3}}+\ldots, \tag{2.3.10}$$

$$\rho \sim r+r_1/k^{\frac{2}{3}}+\ldots \tag{2.3.11}$$

and

$$\sigma \sim s+s_1/k^{\frac{2}{3}}+\ldots, \tag{2.3.12}$$

where $A_j$, $B_j$, $r_j$ and $s_j$ are all independent of $k$.

By substituting these expansions back into the equations and boundary conditions and extracting like powers of $k$, noting that $\mathrm{Ai}\,(k^{\frac{2}{3}}\rho)$ and $\mathrm{Ai}'\,(k^{\frac{2}{3}}\rho)$ need to be expanded first by asymptotic expansions to show that $r=0$ on the boundary and then by Taylor's series, we obtain the following sequence of equations:

$$(\nabla s)^2-r(\nabla r)^2=N^2, \tag{2.3.13}$$

$$\nabla r\,.\,\nabla s=0, \tag{2.3.14}$$

on $S$

$$r=0, \tag{2.3.15}$$

$$2r\nabla r\,.\,\nabla r_1+r_1(\nabla r)^2-2\nabla s\,.\,\nabla s_1=0, \tag{2.3.16}$$

$$\nabla s\,.\,\nabla r_1+\nabla r\,.\,\nabla s_1=0, \tag{2.3.17}$$

on $S$

$$\frac{\partial r}{\partial n}\mathrm{Ai}'(r_1)+\zeta\mathrm{Ai}(r_1)=0, \tag{2.3.18}$$

$$\nabla\,.\,(A_0\nabla s)+\nabla s\,.\,\nabla A_0+\nabla\,.\,(B_0 r\nabla r)+r\nabla r\,.\,\nabla B_0=0, \tag{2.3.19}$$

$$\nabla\,.\,(A_0\nabla r)+\nabla r\,.\,\nabla A_0-\nabla\,.\,(B_0\nabla s)-\nabla s\,.\,\nabla B_0=0, \tag{2.3.20}$$

on $S$

$$A_0\,\mathrm{Ai}(r_1)\frac{\partial s_1}{\partial n}+B_0\left(\mathrm{Ai}(r_1)\,r_1\frac{\partial r}{\partial n}+\zeta\mathrm{Ai}'(r_1)\right)=0, \tag{2.3.21}$$

$$2r\nabla r\,.\,\nabla r_2+r_2(\nabla r)^2-2\nabla s\,.\,\nabla s_2=(\nabla s_1)^2-2r_1(\nabla r_1\,.\,\nabla r)-r(\nabla r_1)^2-\frac{A_0\nabla^2A_0+rB_0\nabla^2B_0}{A_0^2+rB_0^2}, \tag{2.3.22}$$

$$\nabla s\,.\,\nabla r_2+r\nabla\,.\,\nabla s_2=-2\nabla r_1\,.\,\nabla s_1-\frac{1}{2}\frac{A_0\nabla^2B_0-B_0\nabla^2A_0}{A_0^2+rB_0^2}, \tag{2.3.23}$$

on $S$

$$A_0r_2\left(\mathrm{Ai}(r_1)\,r_1\frac{\partial r}{\partial n}+\zeta\mathrm{Ai}'(r_1)\right)=-B_0\frac{\partial s_1}{\partial n}\mathrm{Ai}'(r_1)-\frac{\partial A_0}{\partial n}\mathrm{Ai}(r_1). \tag{2.3.24}$$

Further equations closely resemble equations (2.3.19) to (2.3.24) with progressively more cumbersome 'known' non-homogeneous terms on the right-hand sides. It should be noted that



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the boundary condition (2.3.18) has in general a denumerable number of roots for $r_1$ on $S$, each root corresponding to a different wave mode.

From these equations provided suitable starting values are given, for example $A_0$ known along a constant phase curve on $S$, we can calculate successively $r, s; r_1, s_1; A_0, B_0; r_2, s_2$; etc. Since $r = 0$ on $S$ there is some degeneracy in each pair of equations on $S$ and provided we demand that the functions are always finite then the single explicit boundary condition is sufficient to determine uniquely the two functions. Results are only meaningful if whispering gallery waves can exist which is simply if $\partial r/\partial n > 0$ and can easily be checked in the calculation of $r$ and $s$.

### 2.4. *Waves in a circular region*

One of the simplest problems for which the exact solution of equations (2.1.1) and (2.1.2) is known is waves circulating within a circular region of a homogeneous material with $N = 1$ and on the $u = 0$, i.e. $\zeta = \infty$. The exact solution is

$$u = \exp(\mathrm{i}m\phi)\, J_m(kR), \tag{2.4.1}$$

where $J_m$ is a Bessel function of order $m$, $(R, \phi)$ are the usual cylindrical polar coordinates and to obey the boundary condition on $R = 1$ $m$ must satisfy

$$J_m(k) = 0. \tag{2.4.2}$$

Often a single valued condition is used which restrains $m$ to integer values so equation (2.4.2) becomes an eigenvalue equation for $k$.

Using the asymptotic methods as developed in the previous section it can be shown that in common with other 'exact problems' $B \equiv 0$ in addition the simple geometry enforces the functional dependences $A_j(R)$, $r_j(R)$ and $s_j(\phi)$. These simplifications give from equations (2.3.14 to 2.3.21) the equations

$$\frac{1}{R^2}\left(\frac{\mathrm{d}s}{\mathrm{d}\phi}\right)^2 - r\left(\frac{\mathrm{d}r}{\mathrm{d}R}\right)^2 = 1, \tag{2.4.3}$$

on $R = 1$

$$r = 0, \tag{2.4.4}$$

$$2r\frac{\mathrm{d}r}{\mathrm{d}R}\frac{\mathrm{d}r_1}{\mathrm{d}R} + r_1\left(\frac{\mathrm{d}r}{\mathrm{d}R}\right)^2 - \frac{2}{R^2}\frac{\mathrm{d}s}{\mathrm{d}\phi}\frac{\mathrm{d}s_1}{\mathrm{d}\phi} = 0, \tag{2.4.5}$$

on $R = 1$

$$\mathrm{Ai}\,(r_1) = 0, \tag{2.4.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(RA_0^2\frac{\mathrm{d}r}{\mathrm{d}R}\right) = 0. \tag{2.4.7}$$

It is not difficult to solve these equations to obtain

$$s = \phi, \tag{2.4.8}$$

$$\tfrac{2}{3}r^{\frac{3}{2}} = \int_R^1 R^{-1}(1-R^2)^{\frac{1}{2}}\,\mathrm{d}R, \tag{2.4.9}$$

$$s_1 = r_1(1)\, 2^{-\frac{1}{3}}\phi, \tag{2.4.10}$$

$$r_1 = r_1(1)\, 2^{-\frac{1}{3}} r^{-\frac{1}{2}} \int_R^1 R^{-1}(1-R^2)^{-\frac{1}{2}}\,\mathrm{d}R \tag{2.4.11}$$

and

$$A = C\left(\frac{r}{1-R^2}\right)^{\frac{1}{4}}, \tag{2.4.12}$$

where $C$ is an arbitrary constant.

One way of comparing the exact and asymptotic solutions is by using the uniform asymptotic expansion for Bessel functions of large order

$$m^{\frac{1}{3}} J_m(mz) \sim \left(\frac{4\xi}{1-z^2}\right)^{\frac{1}{2}} \mathrm{Ai}(m\xi^{\frac{2}{3}}), \tag{2.4.13}$$

where

$$\tfrac{2}{3}\xi^{\frac{3}{2}} = \int_z^1 z^{-1}(1-z^2)\,\mathrm{d}z. \tag{2.4.14}$$

The boundary condition (2.4.2) gives when $z = k/m$

$$m^{\frac{2}{3}}\xi \sim a_n, \tag{2.4.15}$$

in which $a_n$ is the $n$th zero of the Airy function. From these relationships it is easy to deduce that

$$k \sim m - a_n 2^{-\frac{1}{3}} m^{\frac{1}{3}}. \tag{2.4.16}$$

It is now straightforward to replace $m$ by $k$ as the large parameter in the asymptotic expansions, in keeping with the direct asymptotic method. It is now easy to show that the direct asymptotic method gives the correct leading terms for $u$. The next approximation has also been checked but demands much more effort than does the above analysis.

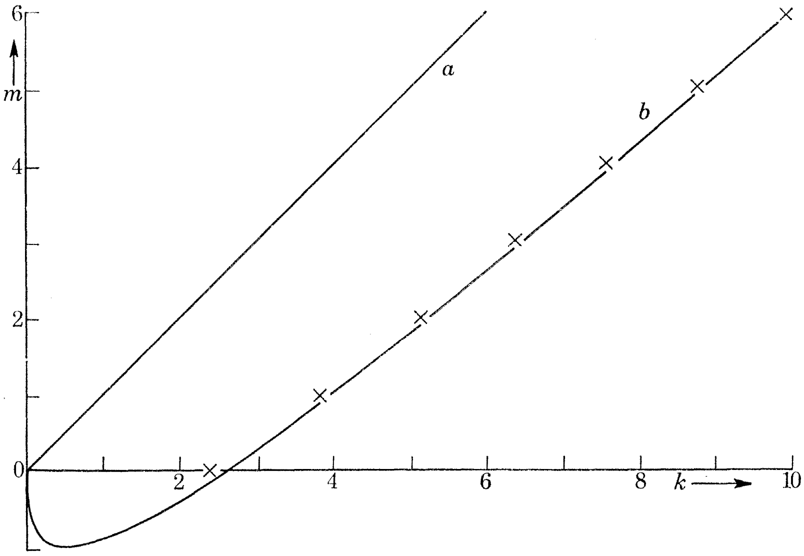


FIGURE 2. Curve $a$, one-term approximation; $b$, two-term approximation. $\times$, Exact results.

In practice the criterion for choosing an approximate method is not its asymptotic correctness but its numerical accuracy at moderate frequencies. A simple means of testing the accuracy in this example is by comparing the exact and asymptotic values for $m$. Figure 2 shows results for the lowest mode calculated from the first approximation

$$m = k \tag{2.4.17}$$

and the second approximation

$$m = k + a_1 2^{-\frac{1}{3}} k^{\frac{1}{3}}, \tag{2.4.18}$$

compared with exact results for the same mode when $m$ takes integer values.

The first approximation is not very useful which is understandable as it is independent of both the value of $\zeta$ and the mode. In contrast the second approximation depends on both the value of $\zeta$ and the mode while its accuracy even at low frequencies is quite remarkable.

### 2.5. *Relation to ray methods*

The physical or ray explanation of the qualitative behaviour of whispering gallery waves involves a family of rays whose caustic is very close to the boundary. The technique developed in this paper produces on mathematical grounds a new sequence of physical quantities $r$, $s$; $A_0$, $B_0$; $r_1$, $s_1$; etc. which are independent of frequency, yet between them give quantitative results for whispering gallery waves. If both the physical explanation and the mathematical solution are valid we must in some manner be able to describe the new physical quantities by the use of rays.

Since the new physical quantities were deduced for $k$ very large the ray interpretation should involve rays whose caustic is arbitrarily close to the boundary which necessarily means rays whose caustic is the boundary. This deduction is readily supported by the changes of variables

$$\phi^{\pm} = s \pm \mathrm{i}\tfrac{2}{3}r^{\frac{3}{2}}, \tag{2.5.1}$$

$$z_0^{\pm} = r^{-\frac{1}{4}}A_0 \mp \mathrm{i}r^{\frac{1}{4}}B_0, \tag{2.5.2}$$

$$\phi_1^{\pm} = s_1 \pm \mathrm{i}r^{\frac{1}{2}}r_1, \tag{2.5.3}$$

etc.

which lead to the sequence of equations

$$(\nabla\phi^{\pm})^2 = N^2, \tag{2.5.4}$$

$$2\nabla\phi^{\pm}.\nabla z_0^{\pm} + z_0^{\pm}\nabla^2\phi^{\pm} = 0, \tag{2.5.5}$$

$$\nabla\phi^{\pm}.\nabla\phi_1^{\pm} = 0, \tag{2.5.6}$$

etc.

Equations (2.5.4) and (2.5.5) can be recognized as being the 'eikonal' and 'transport' equations respectively of geometrical optics. Along ray paths the equations are all ordinary differential equations so can be solved by marching along these directions, however, the directions are complex so in practice the previous set of equations (2.3.13) to (2.3.24) are more convenient to solve.

The combinations of variables used in the transformations (2.5.1) to (2.5.3) arise naturally from the asymptotic solution for $u$, if away from $\rho = 0$ the Airy functions are replaced by their asymptotic expansions of the form

$$\mathrm{Ai}(k^{\frac{2}{3}}\rho) \sim \tfrac{1}{2}\pi^{-\frac{1}{2}}k^{-\frac{1}{6}}\rho^{-\frac{1}{4}}\exp\left(-\tfrac{2}{3}k\rho^{\frac{3}{2}}\right) \tag{2.5.7}$$

and $A$, $B$, $\rho$ and $\sigma$ are also expanded for large $k$. The leading term for $u$ then becomes

$$\tfrac{1}{2}\pi^{-\frac{1}{2}}k^{-\frac{1}{6}}(r^{-\frac{1}{4}}A_0 - \mathrm{i}r^{\frac{1}{4}}B_0)\exp\left(\mathrm{i}k(s+\mathrm{i}\tfrac{2}{3}r^{\frac{3}{2}}) + \mathrm{i}k^{\frac{1}{3}}(s_1 + \mathrm{i}r_1 r^{\frac{1}{2}})\right). \tag{2.5.8}$$

The alternative signs in transformations (2.5.1) to (2.5.3) arise from the solution for $u$ which corresponds to whispering gallery waves propagating in the opposite direction to that previously considered:

$$u = \exp(-\mathrm{i}k\sigma)\{A\,\mathrm{Ai}(k^{\frac{2}{3}}\rho) - \mathrm{i}k^{-\frac{1}{3}}B\,\mathrm{Ai}'(k^{\frac{2}{3}}\rho)\}. \tag{2.5.9}$$

For eigenvalue problems involving whispering gallery waves, such as the problem solved in § 2.4, Keller & Rubinow (1960) have developed a simple real ray solution. Although their geometric construction is suspect except for certain simple boundary shapes, their results for eigenvalues are identical to first order in $m$ with the results of the present technique and for high modes their results are very accurate to second order also.





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## 3. Ducted waves

### 3.1. *Introduction*

Unlike the other classes of trapped waves examined in this paper ducted waves are not guided by the presence of boundaries but only by the non-homogeneity of the medium in which they propagate, rays being refracted back and forth across an internal surface $S$ as shown in figure 3. For example it is occasionally observed that radio signals from transmitters on the Earth are received after a very long delay, usually between 3 and 15 s. This is due to waves which are guided along elevated ducts far beyond the Earth, the ducts being caused by non-homogeneity in the belts of ions trapped by the Earth's magnetic field. Another example, the trapping of long water waves over an oceanic ridge, is the subject of § 3.5.

In order to avoid possibly misleading details we shall first restrict our attention to the representative problem of scalar ducted waves $u(\mathbf{x})$ which satisfy the reduced wave equation

$$\nabla^2 u + k^2 N^2 u = 0, \tag{3.1.1}$$

in which $N$ is the refractive index of the medium and $k$ is the frequency of the waves. The doubly infinite domain of ducted wave problems seems to present great obstacles to exact analytical methods and even for this most simple type of ducted wave problem there are no solutions available for continuous media.

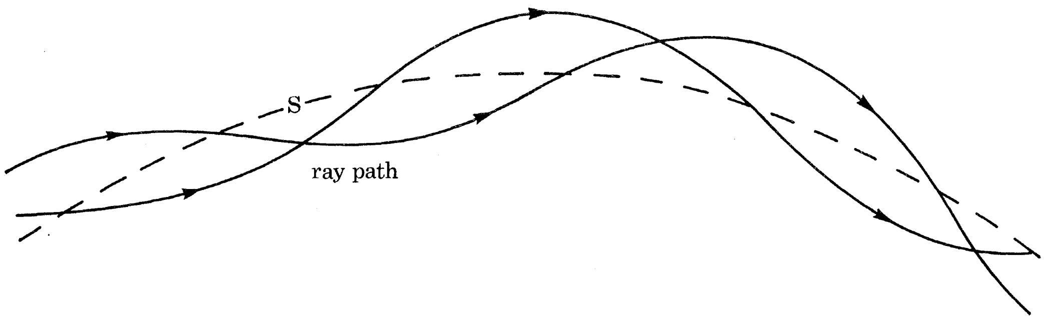


Figure 3

### 3.2. *Fundamental solution for ducted waves*

The argument given in the first paragraph of § 2.2 applies equally well to ducted waves, thus in order to find the 'fundamental solution' for ducted waves we need only study problems in which the internal surface $S$ is perfectly flat and the refractive index $N$ is a function only of the perpendicular distance $x$ from $S$. It is of course essential that $N$ attains its maximum value on $S$.

We seek solutions of equation (3.1.1) corresponding to ducted waves propagating along $S$.

$$u = \hat{u}(x) \exp(ikly). \tag{3.2.1}$$

By substituting this representation for $u$ into equation (3.1.1) together with the trapped wave requirement that $u$ tends to zero far from $S$, we get an eigenvalue problem for $\hat{u}$ and $l$:

$$\frac{d^2\hat{u}}{dx^2} + k^2(N^2 - l^2)\,\hat{u} = 0 \tag{3.2.2}$$

and as $x \to \pm\infty$

$$\hat{u} \to 0. \tag{3.2.3}$$

The local form of solutions near $x = 0$ as $k \to \infty$ can be found by making the transformations

$$\xi = k^{\frac{1}{2}}x \tag{3.2.4}$$

and

$$l = N(0) + k^{-1}\hat{\ell}. \tag{3.2.5}$$

Setting $k = \infty$ now yields the equation

$$\frac{d^2 u_\infty}{d\xi^2} + \left(N(0)\left.\frac{d^2 N}{dx^2}\right|_0 \xi^2 - 2N(0)\,\hat{\ell}_\infty\right)\hat{u}_\infty = 0,$$

which has bounded solutions:

$$\hat{u}_\infty(\xi) = C\Psi_n(\alpha\xi), \tag{3.2.6}$$

$$\hat{\ell}_\infty = -(n+\tfrac{1}{2})\,\beta \tag{3.2.7}$$

where

$$\alpha = \left(-N(0)\left.\frac{d^2N}{dx^2}\right|_0\right)^{-\frac{1}{4}}, \tag{3.2.8}$$

$$\beta = \left(-\frac{1}{N(0)}\left.\frac{d^2N}{dx^2}\right|_0\right)^{\frac{1}{2}}. \tag{3.2.9}$$

$C$ is an arbitrary constant and $\Psi_n$ is a Hermite function of positive integer order $n$.

This local solution defines the 'fundamental solution' for ducted waves. In the notation of § 1.2

$$f_1(t) = \Psi_n(t) \tag{3.2.10}$$

and $f_1$ satisfies the ordinary differential equation

$$d^2f_1/dt^2 + ((2n+1) - t^2)f_1 = 0, \tag{3.2.11}$$

therefore

$$f_2(t) = \Psi_n'(t). \tag{3.2.12}$$

$\Psi_n$ is related to the Hermite polynomial of degree $n$ by

$$\Psi_n(t) = \exp\left(-\tfrac{1}{2}t^2\right) H_n(t). \tag{3.2.13}$$

It should be noted that this 'fundamental solution' is an exact solution of equation (3.1.1) for the refractive index

$$\left(N^2(0) + x^2 N(0)\left.\frac{d^2N}{dx^2}\right|_0\right)^{\frac{1}{2}}, \tag{3.2.14}$$

which does not correspond to any physically meaningful situation since it becomes complex when $x$ is large.

### 3.3. *Derivation of equations*

In order to find ducted wave solutions of equation (3.1.1) when $S$ is not necessarily flat we again follow the general principles given in § 1.2. From the 'fundamental solution' we deduce a general solution for ducted waves

$$u = \exp(ik\sigma)\,\{A\Psi_n(k^{\frac{1}{2}}\rho) + ik^{-\frac{1}{2}}B\Psi_n'(k^{\frac{1}{2}}\rho)\}, \tag{3.3.1}$$

in which $A$, $B$, $\rho$, $\sigma$ are all real slowly varying functions and the $k^{-\frac{1}{2}}$ factor in the second term is found from the requirement that when $u$ is substituted into the equation (3.1.1) the $A$ and $B$ terms are of the same order in $k$.

By substituting the assumed general solution (3.3.1) directly into the reduced wave equation (3.1.1) using Hermite's equation (3.2.11) to replace $\Psi_n''$ in terms of $\Psi_n$, we obtain

$$\begin{aligned}
&k^2\Psi_n\{A[\rho^2(\nabla\rho)^2 - (\nabla\sigma)^2 + N^2] - 2\rho^2 B\nabla\rho\,.\,\nabla\sigma\} + k^{\frac{3}{2}}\Psi_n'\{+2iA\nabla\rho\,.\,\nabla\sigma + iB[\rho^2(\nabla\rho)^2 - (\nabla\sigma)^2 + N^2]\}\\
&\quad + k\Psi_n\{-(2n+1)\,A(\nabla\rho)^2 - 2B\nabla\rho\,.\,\nabla\sigma + i[\nabla\,.\,(A\nabla\sigma) + \nabla\sigma\,.\,\nabla A + \nabla\,.\,(B\rho^2\nabla\rho) + \rho^2\nabla\rho\,.\,\nabla B]\}\\
&\quad + k^{\frac{1}{2}}\Psi_n'\{\nabla\,.\,(A\nabla\rho) + \nabla\rho\,.\,\nabla A - \nabla\,.\,(B\nabla\sigma) - \nabla\sigma\,.\,\nabla B - (2n+1)\,iB(\nabla\rho)^2\}\\
&\quad + \Psi_n\{\nabla^2 A - (2n+1)\,i\nabla\,.\,(B\nabla\rho) + \nabla\rho\,.\,\nabla B\} + k^{-\frac{1}{2}}\Psi_n'\{i\nabla^2 B\} = 0.
\end{aligned} \tag{3.3.2}$$

As explained in § 1.2 because $\Psi_n$ and $\Psi'_n$ are independent functions with derivatives of $O(k^{\frac{1}{2}})$ while $A$, $B$, $\rho$, $\sigma$ all vary slowly we must set the coefficients of $\Psi_n$ and $\Psi'_n$ separately to zero. Further in this context as $A$, $B$, $\rho$, $\sigma$ are all real we have a further doubling-up of equations when the real and imaginary terms are isolated. The resulting four equations are most conveniently expressed

$$\rho^2(\nabla\rho)^2-(\nabla\sigma)^2+N^2-\frac{(2n+1)}{k}\left((\nabla\rho)^2+\frac{2\nabla\rho\,.\,\nabla\sigma AB}{A+\rho^2B}\right)+\frac{1}{k^2}\,\frac{A\nabla^2A+\rho^2B\nabla^2B}{A^2+\rho^2B^2}=0, \tag{3.3.3}$$

$$\nabla\rho\,.\,\nabla\sigma+\frac{(2n+1)}{k}\,\frac{\nabla\rho\,.\,\nabla\sigma B^2}{A^2+\rho^2B^2}+\frac{1}{k^2}\,\frac{A\nabla^2B-B\nabla^2A}{2(A^2+\rho^2B^2)}=0, \tag{3.3.4}$$

$$\nabla\,.\,(A\nabla\sigma)+\nabla\sigma\,.\,\nabla A+\nabla\,.\,(B\rho^2\nabla\rho)+\rho^2\nabla\rho\,.\,\nabla B-\frac{(2n+1)}{k}(\nabla\,.\,(B\nabla\rho)+\nabla\rho\,.\,\nabla B)=0, \tag{3.3.5}$$

and

$$\nabla\,.\,(A\nabla\rho)+\nabla\rho\,.\,\nabla A-\nabla\,.\,(B\nabla\sigma)-\nabla\sigma\,.\,\nabla B=0. \tag{3.3.6}$$

From the above four equations we deduce that as $k\to\infty$, $A$, $B$, $\rho$, $\sigma$ must have the following asymptotic expansions:

$$A\sim A_0+A_1/k+\ldots, \tag{3.3.7}$$

$$B\sim B_0+B_1/k+\ldots, \tag{3.3.8}$$

$$\rho\sim r+r_1/k+\ldots, \tag{3.3.9}$$

$$\sigma\sim s+s_1/k+\ldots \tag{3.3.10}$$

where $A_j$, $B_j$, $r_j$, $s_j$ are all independent of $k$.

By substituting these expansions back into the equations (3.3.3) to (3.3.6) and then extracting like powers of $k$, we obtain the following sequence of equations:

$$(\nabla s)^2-r^2(\nabla r)^2=N^2, \tag{3.3.11}$$

$$\nabla s\,.\,\nabla r=0, \tag{3.3.12}$$

$$\nabla\,.\,(A_0\nabla s)+\nabla s\,.\,\nabla A_0+\nabla\,.\,(B_0r^2\nabla r)+r^2\nabla r\,.\,\nabla B_0=0, \tag{3.3.13}$$

$$\nabla\,.\,(A_0\nabla r)+\nabla r\,.\,\nabla A_0-\nabla\,.\,(B_0\nabla s)-\nabla s\,.\,\nabla B_0=0, \tag{3.3.14}$$

$$r^2\nabla r\,.\,\nabla r_1+r_1r(\nabla r)^2-\nabla s\,.\,\nabla s_1=(n+\tfrac{1}{2})\,(\nabla r)^2, \tag{3.3.15}$$

$$\nabla s\,.\,\nabla r_1+\nabla r\,.\,\nabla s_1=0, \tag{3.3.16}$$

etc.

Further equations resemble equations (3.3.13) to (3.3.16) with progressively larger non-homogeneous right-hand sides.

From these equations, provided that suitable starting conditions are given somewhere on $S$, we can solve successively for $r$, $s$; $A_0$, $B_0$; $r_1$, $s_1$; etc. Since $r=0$ on $S$ each pair of equations becomes extremely degenerate on $S$ and although there are no explicit boundary conditions provided we demand that the functions are always finite then the two functions are uniquely specified. Another important feature is that in general the surface $S$ is not the surface of maximum refractive index. One albeit crude method of finding this surface is by following several ray paths to find successive approximations for $S$.

### 3.4. *Relations to other methods*

An alternative asymptotic solution for ducted waves satisfying the reduced wave equation (3.1.1) can be deduced from the general solution (3.3.1) if we can assume that $\rho$ and $\sigma$ are independent of $k$ and permit $A$ and $B$ to be complex.



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The two dominant terms in equation (3.3.2) vanish identically if

$$\rho^2(\nabla\rho)^2-(\nabla\sigma)^2+N^2 = 0 \tag{3.4.1}$$

and
$$\nabla\rho\,.\,\nabla\sigma = 0. \tag{3.4.2}$$

From the remaining terms, setting the coefficients of $\Psi_n$ and $\Psi'_n$ separately to zero, we obtain the two equations

$$\nabla\,.\,(A\nabla\sigma)+\nabla\sigma\,.\,\nabla A+\nabla\,.\,(B\rho^2\nabla\rho)+\rho^2\nabla\rho\,.\,\nabla B$$
$$+(2n+1)\,\mathrm{i}A(\nabla\rho)^2-(1/k)\,(\mathrm{i}\nabla^2A+(2n+1)\,(\nabla\,.\,(B\nabla\rho)+\nabla\rho\,.\,\nabla B)) = 0 \tag{3.4.3}$$

and
$$\nabla\,.\,(A\nabla\rho)+\nabla\rho\,.\,\nabla A-\nabla\,.\,(B\nabla\sigma)-\nabla\sigma\,.\,\nabla B-(2n+1)\,\mathrm{i}B(\nabla\rho)^2+(\mathrm{i}/k)\,\nabla^2B = 0. \tag{3.4.4}$$

When $k$ is large $A$ and $B$ must possess asymptotic expansions of the forms

$$A \sim A_0+A_1/(k)+\ldots. \tag{3.4.5}$$

and
$$B \sim B_0+B_1/(k)+\ldots, \tag{3.4.6}$$

where $A_j$ and $B_j$ are all independent of $k$.

By substituting these expansions into the equations (3.4.3) and (3.4.4) then extracting like-powers of $k$ we obtain the following sequence of equations

$$\nabla\,.\,(A_0\nabla\sigma)+\nabla\sigma\,.\,\nabla A_0+\nabla\,.\,(B_0\rho^2\nabla\rho)+\rho^2\nabla\rho\,.\,\nabla B_0+(2n+1)\,\mathrm{i}A_0(\nabla\rho)^2 = 0, \tag{3.4.7}$$

$$\nabla\,.\,(A_0\nabla\rho)+\nabla\rho\,.\,\nabla A_0-\nabla\,.\,(B_0\nabla\sigma)-\nabla\sigma\,.\,\nabla B_0-(2n+1)\,\mathrm{i}B_0(\nabla\rho)^2 = 0, \tag{3.4.8}$$

etc.

Further equations closely resemble the above two equations except in the presence of non-homogeneous terms on the right-hand sides.

In one sense this alternative asymptotic solution is more efficient than the one derived in the previous subsection since in order to obtain a uniformly valid solution for $u$ in a finite region as $k\to\infty$ it is only necessary to know $A_0$, $B_0$, $\rho$ and $\sigma$ while the previous solution requires $r$, $s$, $r_1$, $s_1$, $A_0$ and $B_0$ (where the values of $A_0$, $B_0$ are different in the two solutions). However, the previous solution makes explicit allowances for the variation with $k$ of both the decay rate from $S$ and the velocity along $S$ while the alternative solution makes these adjustments implicitly with consequent reduced reliability at low frequencies.

The relations between the coefficients in the two solutions can be found by applying Taylor's series to the previous asymptotic solution when $k$ is large. For example in terms of the first solution the amplitude factors $A_0$ and $B_0$ in the alternative solution are respectively

$$\exp\,(\mathrm{i}s_1)\,(A_0+\mathrm{i}r_1r^2B_0) \tag{3.4.9}$$

and
$$\exp\,(\mathrm{i}s_1)\,(B_0-\mathrm{i}r_1A_0). \tag{3.4.10}$$

In the ray explanation of ducted waves there is a narrow section of width $O(k^{-\frac{1}{2}})$ between two caustics in which there are real rays and beyond these caustics there are two separate regions of complex rays. The asymptotic solutions produce new physical quantities which although independent of $k$ can between them describe ducted waves when $k$ is very large. Correspondingly a ray interpretation of the new physical quantities should involve rays whose caustics are arbitrarily close together so that all points not on $S$ lie in one of the two complex ray regions.

With this physical motivation it is not difficult to recognize how to combine the equations (3.3.11) to (3.3.16) together into a form strongly resembling geometrical optics.

$$(\nabla\phi^{\pm})^2 = N^2, \tag{3.4.11}$$

$$2\nabla\phi^{\pm}.\nabla z_0^{\pm} + z_0^{\pm}\nabla^2\phi^{\pm} = 0, \tag{3.4.12}$$

$$\nabla\phi^{\pm}.\nabla\phi_1^{\pm} = -(n+\tfrac{1}{2})(\nabla r)^2, \tag{3.4.13}$$

etc.;

where $\phi^{\pm}$, $z_0^{\pm}$ and $\phi_1^{\pm}$ are defined by

$$\phi^{\pm} = s \pm \mathrm{i}r^2/2, \tag{3.4.14}$$

$$x_0^{\pm} = r^{-\frac{1}{2}}A_0 \mp \mathrm{i}r^{\frac{1}{2}}B_0 \tag{3.4.15}$$

and

$$\phi_1^{\pm} = s_1 \pm \mathrm{i}rr_1. \tag{3.4.16}$$

Equations (3.4.11) and (3.4.12) are respectively the 'eikonal' and 'transport' equations of geometrical optics. Along the ray paths all the higher equations become linear ordinary differential equations so can be solved by marching along these directions though in practice as these directions are complex the original set of equations (3.3.11) to (3.3.16) is more useful.

### 3.5. *Long waves trapped over an oceanic ridge*

Long water waves are usually regarded as low-frequency waves, which indeed they are when compared with the more obvious surface waves whose periods are only seconds, or even when compared with frequencies based upon the depth of the water since by definition the wavelength of a long wave greatly exceeds the local depth. However, the only relevant comparison scale is based upon the horizontal size of topographical changes and for a typical model of the mid-Atlantic ridge, as shown in figure 4, a lowest mode wave with a period of 30 min has a high frequency. The effects of the Earth's rotation might not be negligible for waves of such periods so we shall include the Coriolis force which has an asymmetric effect on wave propagation.

Let $\eta$ denote the elevation of the free surface above its equilibrium level, $\boldsymbol{v}$ denote the mean horizontal velocity averaged throughout the depth, $\boldsymbol{f}$ the vertical Coriolis vector and $h$ the undisturbed depth of the water. Then if density variations are ignored and if $\eta \ll h$ the momentum and continuity equations for long waves may be linearized to the forms

$$\partial\boldsymbol{v}/\partial t + g\nabla\eta + \boldsymbol{f}\times\boldsymbol{v} = 0 \tag{3.5.1}$$

and

$$\partial\eta/\partial t + \nabla.(h\boldsymbol{v}) = 0. \tag{3.5.2}$$

For waves with time dependence $\exp(-\mathrm{i}\omega t)$ we can eliminate $\boldsymbol{v}$ from the above equations to obtain the single scalar equation

$$\nabla.(h\nabla\eta) + \left(\frac{\omega^2}{g} - \frac{f^2}{g}\right)\eta - \frac{\mathrm{i}}{\omega}\nabla\eta.(\nabla\times(h\boldsymbol{f})) + \frac{h}{\omega^2 - f^2}\nabla f^2.\left(\nabla\eta + \frac{\mathrm{i}}{\omega}\nabla\eta\times\boldsymbol{f}\right) = 0. \tag{3.5.3}$$

In practice the $\nabla f^2$ terms are negligible for high-frequency waves so in order to make algebraic manipulations considerably less tedious we shall ignore these terms.

We can very easily transform this equation into a form much more closely resembling the standard reduced wave equation (3.1.1) by setting $u = \eta h^{\frac{1}{2}}$ and $N^2 = 1/h$. The equation satisfied by $u$ is

$$\nabla^2 u + \frac{\omega^2}{g}N^2 u = u\left[\frac{f^2}{g}N^2 + N\nabla^2(N^{-1}) + \frac{\mathrm{i}}{\omega}\nabla N^2.(\nabla\times N^{-2}\boldsymbol{f})\right] + \frac{2\mathrm{i}}{\omega}N^2\nabla u.(\nabla\times N^{-2}\boldsymbol{f}). \tag{3.5.4}$$

From this equation it is clear that when $\omega$ is large the solutions will behave very similarly to the solutions of equation (3.1.1), for example the 'fundamental solutions' for the two equations are the same.

Closely paralleling the arguments of § 3.3 from the assumed general solution

$$\eta = \exp(\mathrm{i}k\sigma)\,\{A\Psi_n(k^{\frac{1}{2}}\rho) + \mathrm{i}k^{-\frac{1}{2}}B\Psi_n'(k^{\frac{1}{2}}\rho)\}, \tag{3.5.5}$$

in which $k^2$ replaces $\omega^2/g$, we obtain instead of equations (3.3.3) to (3.3.6) the slightly more complicated equations:

$$h(\rho^2(\nabla\rho)^2 - (\nabla\sigma)^2) + 1 - \frac{(2n+1)h}{k}\left((\nabla\rho)^2 + \frac{2\nabla\rho\,.\,\nabla\sigma AB}{A^2+\rho^2B^2}\right)$$
$$+\frac{1}{k^2}\left(+g^{-\frac{1}{2}}\nabla\sigma\,.\,(\nabla\times hf) - \frac{f^2}{g} + \frac{A\nabla\,.\,(h\nabla A) + \alpha^2 B\nabla\,.\,(h\nabla B)}{A^2+\rho^2B^2}\right) - \frac{1}{k^3}\,\frac{g^{-\frac{1}{2}}AB\nabla\rho\,.\,(\nabla\times hf)}{(A^2+\rho^2B^2)} = 0, \tag{3.5.6}$$

$$h\nabla\rho\,.\,\nabla\sigma + \frac{(2n+1)\,h}{k}\,\frac{\nabla\rho\,.\,\nabla\sigma B^2}{A^2+\rho^2B^2} + \frac{1}{k^2}\left(-g^{-\frac{1}{2}}\nabla\rho\,.\,(\nabla\times hf) + \frac{A\nabla\,.\,(h\nabla B) - B\nabla\,.\,(h\nabla A)}{2(A^2+\rho^2B^2)}\right)$$
$$+\frac{1}{k^3}\,\frac{(2n+1)\,g^{-\frac{1}{2}}B^2\nabla\rho\,.\,(\nabla\times hf)}{2(A^2+\rho^2B^2)} = 0, \tag{3.5.7}$$

$$\nabla\,.\,(hA\nabla\sigma) + h\nabla\sigma\,.\,\nabla A + \nabla\,.\,(hB\rho^2\nabla\rho) + h\rho^2\nabla\rho\,.\,\nabla B - \frac{(2n+1)}{k}(\nabla\,.\,(hB\nabla\rho) + h\nabla\rho\,.\,\nabla B)$$
$$-\frac{1}{k^2}g^{-\frac{1}{2}}\nabla A\,.\,(\nabla\times hf) = 0, \tag{3.5.8}$$

$$\nabla\,.\,(Ah\nabla\rho) + h\nabla\rho\,.\,\nabla A - \nabla\,.\,(hB\nabla\sigma) - h\nabla\sigma\,.\,\nabla B + \frac{1}{k^2}g^{-\frac{1}{2}}\nabla B\,.\,(\nabla\times hf) = 0. \tag{3.5.9}$$

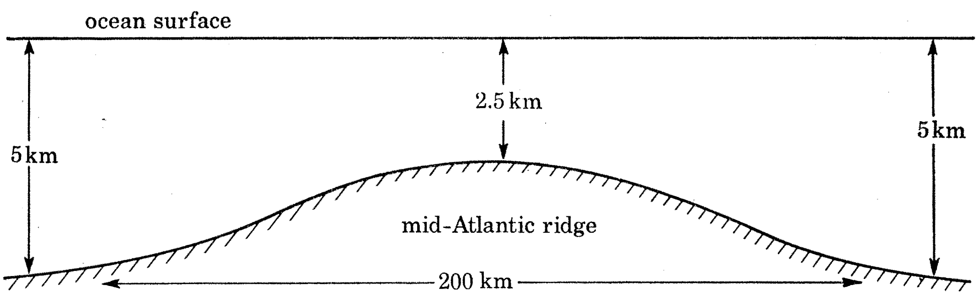


FIGURE 4

We can deduce that as $k \to \infty$, $A$, $B$, $\rho$ and $\sigma$ have asymptotic expansions of the same form as in equations (3.3.7) to (3.3.10). By substituting these expansions into the above equations and then extracting like powers of $k$ we can obtain the equations satisfied by $A_j$, $B_j$, $r_j$ and $s_j$. The final equations can then be solved numerically, subject to the same difficulties as occur for the simplest example of ducted waves.

The Coriolis force affects neither the first nor the second terms in any of the expansions so its asymmetrizing effect is deferred to the third terms. Although the asymptotic solutions have a limited range of application this observation suggests that for models of the mid-Atlantic ridge neither the amplitude nor the phase are significantly affected by the Coriolis force unless the period exceeds several hours. Buchwald (1968) notes the surprisingly small effects of the Coriolis force on waves trapped over a class of straight oceanic ridges.

## 4. Edge waves

### 4.1. *Introduction*

Edge waves are a class of water waves which because of the increasing depth away from a shoreline are trapped near the coast and propagate along it. They were predicted theoretically by Stokes (1846), but since they do not break at the shoreline and are exponentially small away from the shore they were, for many decades, regarded as a mathematical curiosity. From wave measurements considerable effort is required to distinguish the effects of edge waves from the much greater effects of other waves yet Munk, Snodgrass & Gilbert (1964) conclusively showed the existence of edge waves off the California coast.

This insignificance of edge waves is due to the difficulties experienced in exciting an edge wave. To get a significant and coherent wave only one edge wave mode must be excited strongly. If wind-stress provides the forcing this could be done if the speed along the coast of a squall-line equalled the speed of that one mode for a considerable distance. The edge wave would grow, since being a trapped wave very little energy would be radiated away, until the forcing stopped. Donn & Ewing (1956) and Donn (1959) identified destructive waves along the shorelines of the Great Lakes with edge waves generated by travelling squall lines. Thus far from being a mathematical curiosity edge waves are of great importance in the difficult and essential tasks of flood prediction and flood protection.

The mathematical problem, as first solved by Stokes for a uniformly sloping beach, requires a solution of Laplace's equation in the entire water region contained between the bottom and the free surface. Extremely few geometries are known for which exact solutions can be found and even for such situations results are limited in their range of application because the Earth's rotation has been ignored. Direct numerical solutions in real situations for this formulation of edge wave propagation seems quite out of the question because of the elliptic nature of Laplace's equation.

It was shown by Eckart (1951) that since in real situations beach slopes are very small, of the order 1:100, edge waves are long waves with wavelengths considerably greater than the depth of water within their effective width from the shore. Thus it is possible to eliminate all vertical variations and except within a few yards of the shore the linearized shallow water equations apply. In particular for waves with time dependence $\exp(-\mathrm{i}\omega t)$ we can obtain the single scalar equation

$$\nabla . (h\nabla\eta) + \left(\frac{\omega^2}{g} - \frac{f^2}{g}\right)\eta - \frac{\mathrm{i}}{\omega}\nabla\eta . (\nabla\times hf) = 0, \qquad (4.1.1)$$

where, as in § 3.5, we have ignored $\nabla f^2$ terms.

The linearity assumption that $\eta \ll h$ is invalid within the last few yards of the shore and consequently equation (4.1.1) is quite wrong at the shore which is demonstrated by the fact that it implies that $\eta$ will in general have a logarithmic singularity there. However, for the special circumstances of edge waves for which nothing exceptional, such as breaking, happens near the shore solutions of equation (4.1.1) give a very accurate description of the wave height everywhere provided of course $\eta$ is not infinite at the shore. Thus for edge waves we have the boundary condition that

$$\text{on } S \quad \eta \text{ is finite} \qquad (4.1.2)$$

where $S$ denotes the particular shore along which the edge waves travel.

It cannot be over-emphasized that the boundary condition (4.1.2) only applies to edge waves, for example when forcing terms are included in equation (4.1.1) other types of waves may also be produced so if we naïvely applied the boundary condition (4.1.2) our results might be meaningless.

This more tractable formulation of edge wave problems has been used to find several enlightening exact analytic solutions for idealized topographies in which all depth contours are parallel straight lines and $f$ is constant. Figure 5 shows the depth profiles used by Ried (1958), Ball (1967) and Mysak (1968) respectively to show the effects of certain topographical features upon edge wave propagation.

The ray paths of shallow water waves satisfy the equation

$$gh(\nabla\psi)^2 = 1 \tag{4.1.3}$$

and can be interpreted as the flight paths of perfectly smooth particles sliding over the 'reciprocal surface' whose depth is $1/h$, at a speed corresponding to the particles having started at rest at zero depth (Eckart 1950). Thus near the shoreline the reciprocal surface falls away very rapidly which means that particles can only escape from the neighbourhood of the shoreline if they ascend the slope almost normally. To the trapping of particles there corresponds the trapping of waves near to the shore namely edge waves. Just as the linearized shallow water wave equations

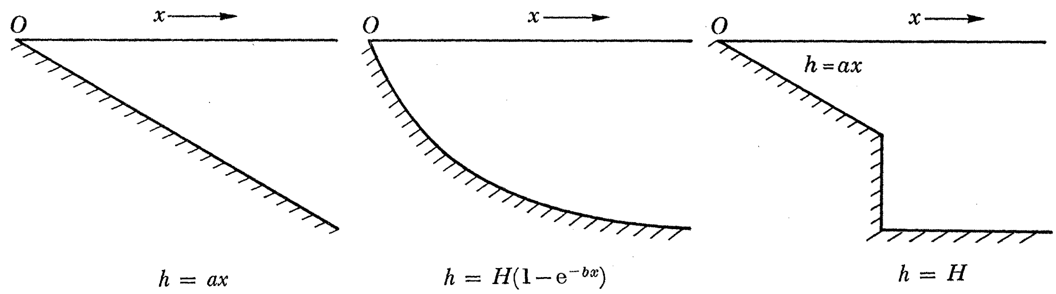


FIGURE 5

are meaningless within a few yards of the shore the analogy breaks down near the shore, since the reciprocal surface falls away so fast that either the particles separate from the reciprocal surface and fall freely under gravity or they never actually reach the shoreline. If this difficulty is ignored it can be shown that a ray and its 'reflexion' form a cusp at the shore, while the rays associated with a particular mode of an edge wave have a caustic away from the shore.

### 4.2. *Fundamental solution for edge waves*

The effective width of a high-frequency edge wave is inversely proportional to $\omega^2$. On this scale the depth profile varies very slowly, in particular the curvature of the coastline becomes negligible. Consequently in order to find the 'fundamental solution' for edge waves it is only necessary to examine problems in which the depth contours are parallel straight lines and $f$ is constant.

In this subsection we shall assume that there is only one coastline $S$ which is perfectly straight and that the depth $h$ is a function only of the perpendicular distance $x$ from $S$. We seek solutions of equations (4.1.1) and (4.1.2) corresponding to edge waves propagating in the $+y$ direction:

$$\eta = \hat{\eta}(x)\exp(\mathrm{i}k^2\hat{\ell}y), \tag{4.2.1}$$

where we have replaced $\omega^2/g$ by $k^2$.

By substituting this representation for $\eta$ into equations (4.1.1) and (4.1.2) together with the trapped wave requirement that $\eta$ tends to zero far from $S$, we get an eigenvalue problem in $\hat{\eta}$ and $\hat{\ell}$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(h\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x}\right)+\left(k^2-k^4\hat{\ell}^2h+kg^{-\frac{1}{2}}f\frac{\mathrm{d}h}{\mathrm{d}x}-\frac{f^2}{g}\right)\hat{\eta} = 0, \tag{4.2.2}$$

$$\text{on} \quad x = 0 \quad h = 0 \quad \text{and} \quad \hat{\eta} \text{ is finite,} \tag{4.2.3}$$

as $x \to \infty$
$$\hat{\eta} \to 0. \tag{4.2.4}$$

The local form of solutions near $x = 0$ as $k \to \infty$ is easily found by making the transformation

$$\xi = k^2 x. \tag{4.2.5}$$

Setting $k = \infty$ now yields the equation

$$\left.\frac{dh}{dx}\right|_0 \frac{d}{d\xi}\left(\xi \frac{d\hat{\eta}_\infty}{d\xi}\right) + \left(1 - \left.\frac{dh}{dx}\right|_0 \hat{\ell}^2_\infty \xi\right)\hat{\eta}_\infty = 0,$$

which has eigenfunctions and eigenvalues:

$$\hat{\eta}_\infty = C\Psi_n(\alpha\xi/(2n+1)) \tag{4.2.6}$$

and
$$\hat{\ell}_\infty = \alpha/(2n+1), \tag{4.2.7}$$

where
$$\alpha = 1\Big/\left.\frac{dh}{dx}\right|_0, \tag{4.2.8}$$

$C$ is an arbitrary constant and $\Psi_n$ is a Laguerre function of positive integer order $n$.

We shall use this local solution to define our 'fundamental solution' for edge waves. In the notation of § 1.2

$$f_1(t) = \psi_n(t/(2n+1)) \tag{4.2.9}$$

and $f_1$ satisfies the ordinary differential equation

$$\frac{d}{dt}\left(t\frac{df_1}{dt}\right) + \left(1 - \frac{t}{(2n+1)^2}\right)f_1 = 0, \tag{4.2.10}$$

where $n$ is a positive integer. We shall define $f_2(t)$:

$$f_2(t) = t\psi_n'(t/(2n+1)). \tag{4.2.11}$$

These rescalings of $f_2$ and the Laguerre functions have been chosen to make all the analysis of the next subsection less cumbersome. $\psi_n$ is related to the Laguerre polynomial of degree $n$ by

$$\psi_n(t) = \exp(-t)\,L_n(2t). \tag{4.2.12}$$

It should be noted that the fundamental solution is only an exact solution to equation (4.1.1) if the Coriolis force is ignored.

### 4.3. *Derivation of equations*

We now return to the general situation where the coastline $S$ is not necessarily straight, there may be other coastlines and the depth of the water may vary in any smooth way. To obtain edge wave solutions of equations (4.1.1) and (4.1.2) we use the 'fundamental solution' to derive the general solution

$$\eta = \exp\left(\frac{ik^2\sigma}{2n+1}\right)\left\{A\Psi_n\left(\frac{k^2\rho}{2n+1}\right) + iB\rho\Psi_n'\left(\frac{k^2\rho}{2n+1}\right)\right\}, \tag{4.3.1}$$

where $A$, $B$, $\rho$, $\sigma$ are all real slowly varying functions which may depend on $k$ and the $k^0$ factor in the second term is found by the requirement that when $\eta$ is substituted into equation (4.1.1) the $A$ and $B$ terms are of the same order in $k$.

By substituting the general solution (4.3.1) directly into the reduced wave equation (4.1.1), using the ordinary differential equation (4.2.10) to replace $\nabla(\rho\psi_n')$ by

$$\nabla\rho\left(\frac{k^2\rho}{(2n+1)} - (2n+1)\right)\psi_n,$$

we obtain the formidable looking equation

$$\frac{k^4\psi_n}{(2n+1)^2}\{Ah[(\nabla\rho)^2-(\nabla\sigma)^2]-2h\rho B\nabla\rho.\nabla\sigma\}+\frac{k^4\rho\psi_n'}{(2n+1)^2}\left\{\mathrm{i}\frac{2h}{\rho}A\nabla\rho.\nabla\sigma+\mathrm{i}hB[(\nabla\rho)^2-(\nabla\sigma)^2]\right\}$$
$$+\frac{k^2\psi_n}{(2n+1)^2}\left\{(2n+1)^2\left[A\left(1-\frac{h}{\rho}(\nabla\rho)^2\right)+2hB\nabla\rho.\nabla\sigma\right]+(2n+1)\,\mathrm{i}[\nabla.(hA\nabla\sigma)+h\nabla\sigma.\nabla A\right.$$
$$\left.+\nabla.(hB\rho\nabla\rho)+h\rho\nabla\rho.\nabla B]\right\}+\frac{k^2\rho\psi_n'}{(2n+1)^2}\left\{(2n+1)\left[\nabla.\left(A\frac{h}{\rho}\nabla\rho\right)+\frac{h}{\rho}\nabla\rho.\nabla A-\nabla(hB\nabla\sigma)-h\nabla\sigma.\nabla B\right]\right.$$
$$\left.+(2n+1)^2\mathrm{i}B\left(1-\frac{h}{\rho}(\nabla\rho)^2\right)\right\}+\frac{k\psi_n}{(2n+1)^2}\{+(2n+1)\,g^{-\frac{1}{2}}(\nabla\times h\boldsymbol{f}).(A\nabla\sigma+B\rho\nabla\rho)\}$$
$$+\frac{k\rho\psi n'}{(2n+1)^2}\left\{-(2n+1)\mathrm{i}g^{-\frac{1}{2}}(\nabla\times h\boldsymbol{f}).\left(\frac{A}{\rho}\nabla\rho-B\nabla\sigma\right)\right\}$$
$$+\frac{\psi_n}{(2n+1)^2}\left\{(2n+1)^2\left(\nabla.(h\nabla A)-\frac{Af^2}{g}\right)-(2n+1)^3\mathrm{i}\nabla.(hB\nabla\rho)+h\nabla\rho.\nabla B\right\}$$
$$+\frac{\rho\psi_n'}{(2n+1)^2}\left\{(2n+1)^2\mathrm{i}\left(\nabla.(h\nabla B)-B\frac{f^2}{g}\right)\right\}+\frac{k^{-1}\psi_n}{(2n+1)^2}\{-(2n+1)^3g^{-\frac{1}{2}}B(\nabla\times h\boldsymbol{f}).\nabla\rho$$
$$-(2n+1)^2\mathrm{i}g^{-\frac{1}{2}}(\nabla\times h\boldsymbol{f}).\nabla A\}+\frac{k^{-1}\rho\psi_n'}{(2n+1)^2}\{+(2n+1)^2g^{-\frac{1}{2}}(\nabla+h\boldsymbol{f}).\nabla B\}=0. \qquad (4.3.2)$$

Following the general principles expounded in § 1.2 we must set the coefficients of $\psi_n$ and $\rho\psi_n'$ separately to zero and then because we have assumed $A$, $B$, $\rho$, $\sigma$ are all real we isolate real and imaginary parts of these coefficients giving in all four equations. The resulting equations are most conveniently expressed:

$$h((\nabla\rho)^2-(\nabla\sigma)^2)+\frac{(2n+1)^2}{k^2}\left(1-\frac{h}{\rho}(\nabla\rho)^2+\frac{2ABh\nabla\rho.\nabla\sigma}{A^2+\rho^2B^2}\right)+\frac{(2n+1)}{k^3}g^{-\frac{1}{2}}(\nabla\times h\boldsymbol{f}).\nabla\sigma+\frac{(2n+1)^2}{k^4}$$
$$\times\left(\frac{A\nabla.(h\nabla A)+\rho^2B\nabla.(h\nabla B)}{A^2+\rho^2B^2}-\frac{f^2}{g}\right)-\frac{(2n+1)^3}{k^5}g^{-\frac{1}{2}}\frac{AB(\nabla\times h\boldsymbol{f}).\nabla\rho}{A^2+\rho^2B^2}=0, \qquad (4.3.3)$$

$$h\nabla\rho.\nabla\sigma-\frac{(2n+1)^2}{2k^2}\frac{\rho B^2h\nabla\rho.\nabla\sigma}{A^2+\rho^2B^2}-\frac{(2n+1)}{2k^3}g^{-\frac{1}{2}}(\nabla\times h\boldsymbol{f}).\nabla\rho+\frac{(2n+1)^2}{2k^4}\rho\frac{A\nabla.(h\nabla B)-B\nabla.(h\nabla A)}{A^2+\rho^2B^2}$$
$$+\frac{(2n+1)^3}{2k^5}g^{-\frac{1}{2}}\frac{\rho^2B^2(\nabla\times h\boldsymbol{f}).\nabla\rho}{A^2+\rho^2B^2}=0, \qquad (4.3.4)$$

$$\nabla.(hA\nabla\sigma)+h\nabla\sigma.\nabla A+\nabla.(hB\rho\nabla\rho)+h\rho\nabla\rho.\nabla B-\frac{(2n+1)^2}{k^2}$$
$$\times(\nabla.(hB\nabla\rho)+h\nabla\rho.\nabla B)-\frac{(2n+1)}{k^3}g^{-\frac{1}{2}}(\nabla\times g\boldsymbol{f}).\nabla A=0 \qquad (4.3.5)$$

and
$$\nabla.\left(\frac{h}{\rho}A\nabla\rho\right)+\frac{h}{\rho}\nabla\rho.\nabla A-\nabla.(hB\nabla\sigma)-h\nabla\sigma.\nabla B+\frac{(2n+1)}{k^3}g^{-\frac{1}{2}}(\nabla\times h\boldsymbol{f}).\nabla B=0. \qquad (4.3.6)$$

We are seeking solutions of the reduced wave equation (4.1.1) which are finite on $S$, if instead we seek the other sequence of solutions which have only logarithmic behaviour near $S$ the only change we need make in the above analysis is to replace $\psi_n$ by its singular counterpart. In order that the singularity is positioned exactly on the shoreline we must demand that

$$\text{on } S \quad \rho=0. \qquad (4.3.7)$$

Near $S$ the new term corresponding to $\rho\psi_n'$ is both non-singular and non-zero so to get only logarithmic behaviour we must also demand that

$$\text{on } S \quad B=0. \qquad (4.3.8)$$

The duality of the singular and regular problems means that these two boundary conditions apply in both problems.

A further boundary condition can be obtained from equation (4.3.3) using the mild requirements that $A$, $B$, $\rho$, $\sigma$ are all slowly varying even on $S$:

$$\text{on } S \quad \lim_{h\to 0}\left\{\left(1-\frac{h}{\rho}(\nabla\rho)^2\right)+\frac{g^{-\frac{1}{2}}(\nabla\times hf)\,.\,\nabla\sigma}{k(2n+1)}+\frac{1}{k^2}\left(\frac{\nabla h\,.\,\nabla A}{A}-\frac{f^2}{g}\right)\right\}=0. \tag{4.3.9}$$

From the equations (4.3.3) to (4.3.6) and most significantly the boundary condition (4.3.9) we deduce that as $k\to\infty$ $A$, $B$, $\rho$ and $\sigma$ must have the following asymptotic expansions,

$$A\sim A_0+A_1/k+\dots, \tag{4.3.10}$$

$$B\sim B_0+B_1/k+\dots, \tag{4.3.11}$$

$$\rho\sim r+r_1/k+\dots \tag{4.3.12}$$

and

$$\sigma\sim s+s_1/k+\dots, \tag{4.3.13}$$

in which $A_j$, $B_j$, $r_j$ and $s_j$ are all independent of $k$.

By substituting these expansions back into the equations and boundary conditions then extracting like-powers of $k$ we obtain the following sequence of equations:

$$(\nabla r)^2-(\nabla s)^2=0, \tag{4.3.14}$$

$$\nabla r\,.\,\nabla s=0, \tag{4.3.15}$$

$$\text{on} \quad S \quad r=0, \tag{4.3.16}$$

$$\text{on} \quad S \quad \lim_{h\to 0}\left(\frac{h}{r}(\nabla r)^2\right)=1, \tag{4.3.17}$$

$$\nabla\,.\,(hA_0\nabla s)+h\nabla s\,.\,\nabla A_0+\nabla\,.\,(hB_0 r\nabla r)+hr\nabla r\,.\,\nabla B_0=0, \tag{4.3.18}$$

$$\nabla\,.\left(\frac{h}{r}A_0\nabla r\right)+\frac{h}{r}\nabla r\,.\,\nabla A_0-\nabla\,.\,(hB_0\nabla s)-h\nabla s\,.\,\nabla B_0=0, \tag{4.3.19}$$

$$\text{on} \quad S \quad B_0=0, \tag{4.3.20}$$

$$\nabla r\,.\,\nabla r_1-\nabla s\,.\,\nabla s_1=0, \tag{4.3.21}$$

$$\nabla r\,.\,\nabla s_1+\nabla s\,.\,\nabla r_1=0, \tag{4.3.22}$$

$$\text{on} \quad S \quad r_1=0, \tag{4.3.23}$$

$$\text{on} \quad S \quad \lim_{h\to 0}\left(-\frac{2h}{r}\nabla r\,.\,\nabla r_1+\frac{hr_1}{r^2}(\nabla r)^2\right)=\lim_{h\to 0}-\frac{g^{-\frac{1}{2}}(\nabla\times hf)\,.\,\nabla s}{(2n+1)}, \tag{4.3.24}$$

etc.

Further equations resemble equations (4.3.18) to (4.3.24) but have progressively larger non-homogeneous terms on the right-hand sides.

There is an obvious difference between the coordinate equations (4.3.21) to (4.3.24) and the amplitude equations (4.3.18) to (4.3.20), namely that the first group are regular and need two boundary conditions to uniquely specify $r_1$ and $s_1$, while the second group are degenerate on $S$ and only need one boundary condition to specify uniquely the non-singular solutions for $A_0$ and $B_0$. However, by a simple change of variables

$$A_0=\left(\frac{r}{h}\right)^{\frac{1}{2}}a_0 \tag{4.3.25}$$

and
$$B_0 = \left(\frac{r}{h}\right)^{\frac{1}{2}} \frac{b_0}{r} \tag{4.3.26}$$
the amplitude equations take the regular form
$$a_0 \nabla^2 s + 2\nabla s . \nabla a_0 + b_0 \nabla^2 r + 2\nabla r . \nabla b_0 = 0, \tag{4.3.27}$$
$$a_0 \nabla^2 r + 2\nabla r . \nabla a_0 - b_0 \nabla^2 s - 2\nabla s . \nabla b_0 = 0, \tag{4.3.28}$$
$$\text{on} \quad S \quad b_0 = 0 \tag{4.3.29}$$
and
$$\text{on} \quad S \quad \partial b_0/\partial n = 0. \tag{4.3.30}$$

From the final regular sequence of equations once initial conditions are given we can solve successively for $r$, $s$; $a_0$, $b_0$; $r_1$, $s_1$, etc. The form of the equations appears to be ideally suited to numerical solutions in real situations. The most obvious procedure being first to solve along the shoreline and then to solve along lines progressively further out from the shore: the values along a line provide values for the derivatives along that line, through the equations we can then obtain the derivatives perpendicular to that line and hence we can evaluate the functions along the next line out from the shore.

### 4.4. *Relations to other methods*

When the Coriolis force is neglected an alternative asymptotic solution for edge waves can be deduced from the general solution (4.3.1) if we assume that $\rho$ and $\sigma$ are independent of $k$ and we permit $A$ and $B$ to be complex.

The two dominant terms in equation (4.3.2) vanish identically if
$$(\nabla\rho)^2 - (\nabla\sigma)^2 = 0 \tag{4.4.1}$$
and
$$\nabla\rho . \nabla\sigma = 0. \tag{4.4.2}$$
From the remaining terms, setting the coefficients of $\psi_n$ and $\rho\psi_n'$ separately to zero, we obtain the two equations
$$\nabla . (hA\nabla\sigma) + h\nabla\sigma . \nabla A + \nabla . (hB\rho\nabla\rho) + h\rho\nabla\rho . \nabla B - (2n+1)\mathrm{i}A\left(1 - \frac{h}{\rho}(\nabla\rho)^2\right)$$
$$- \frac{1}{k^2}((2n+1)^2(\nabla . (hB\nabla\rho) + h\nabla\rho . \nabla B) + (2n+1)\mathrm{i}\nabla . (h\nabla A)) = 0, \tag{4.4.3}$$
and
$$\nabla . \left(A\frac{h}{\rho}\nabla\rho\right) + \frac{h}{\rho}\nabla\rho . \nabla A - \nabla . (hB\nabla\sigma) - h\nabla\sigma . \nabla B + (2n+1)\mathrm{i}B\left(1 - \frac{h}{\rho}(\nabla\rho)^2\right)$$
$$+ \frac{(2n+1)}{k^2}\mathrm{i}(\nabla . (h\nabla B)) = 0. \tag{4.4.4}$$

The boundary conditions on $S$ can once more be deduced by means of the duality between the singular and non-singular problems. Thus we have
$$\text{on} \quad S \quad \lim_{h\to 0} \frac{h}{\rho}(\nabla\rho)^2 = 1 \tag{4.4.5}$$
and
$$\text{on} \quad S \quad B = 0. \tag{4.4.6}$$

When $k$ is large it can be easily shown that $A$ and $B$ must possess asymptotic expansions of the forms
$$A \sim A_0 + A_1/(k^2) + \dots \tag{4.4.7}$$
and
$$B \sim B_0 + B_1/(k^2) + \dots. \tag{4.4.8}$$

By substituting these expansions into the equations (4.4.3) to (4.4.6) and then extracting like-powers of $k$ we obtain the following sequence of equations

$$\nabla.(hA_0\nabla\sigma)+h\nabla\sigma.\nabla A_0+\nabla.(hB_0\rho\nabla\rho)+h\rho\nabla\rho.\nabla B_0-(2n+1)\,\mathrm{i}A_0\left(1-\frac{h}{\rho}(\nabla\rho)^2\right)=0, \quad (4.4.9)$$

$$\nabla.\left(A_0\frac{h}{\rho}\nabla\rho\right)+\frac{h}{\rho}\nabla\rho.\nabla A_0-\nabla.(hB_0\nabla\sigma)-h\nabla\sigma.\nabla B_0+(2n+1)\,\mathrm{i}B_0\left(1-\frac{h}{\rho}(\nabla\rho)^2\right)=0, \quad (4.4.10)$$

etc.

Further equations closely resemble the above two equations except by the presence of non-homogeneous terms on the right-hand sides.

This alternative solution cannot be extended to include the effects of the Coriolis force because of the requirements that both $A$ and $B$ are finite on $S$ (cf. equation (4.3.9)). Thus the alternative solution can only be used when the frequency is high, yet in order that its correction terms of order $k^{-1}$ dominate the Coriolis effects of $k^{-2}$ it is necessary that the frequency is not too high. On its own, the leading term of the alternative solution provides a uniformly valid approximation for $\eta$ as $k\to\infty$ and in this one respect the alternative solution is the more efficient solution.

There is a physical explanation of edge wave behaviour in terms of rays. By paralleling the successful ray interpretations of the asymptotic solutions for whispering gallery waves and ducted waves we can replace equations (4.3.14) to (4.3.24) by the equivalent set

$$(\nabla\phi^{\pm})^2=0, \quad (4.4.11)$$

$$z_0^{\pm}\nabla^2\phi^{\pm}+2\nabla z_0^{\pm}.\nabla\phi^{\pm}=0, \quad (4.4.12)$$

$$\nabla\phi^{\pm}.\nabla\phi_1^{\pm}=0, \quad (4.4.13)$$

etc.,

where $\phi^{\pm}$, $z_0^{\pm}$ and $\phi_1^{\pm}$ are defined by the transformations

$$\phi^{\pm}=s\pm\mathrm{i}r, \quad (4.4.14)$$

$$z_0^{\pm}=a_0\mp\mathrm{i}b_0 \quad (4.4.15)$$

and
$$\phi_1=s_1\pm\mathrm{i}r_1. \quad (4.4.16)$$

There is one unsurmountable difficulty in relating $\phi^{\pm}$ to the complex rays of geometrical optics namely that equation (4.4.11) does not have the refractive index $1/h$ on its right-hand side.

Although the interpretation of the equations (4.4.11) to (4.4.13) is not clear they have many of the properties associated with the geometrical optics' equations. For example, along the complex directions $\phi^{\pm}$ the equations are all ordinary differential equations so can be solved by marching along these directions. However, because the directions are complex, in practice the original set of equations (4.3.14) to (4.3.24) are more convenient to solve.

### 4.5 *Discussion of results*

It is of course essential that in those few situations for which exact analytic solutions are known the asymptotic expansions of the exact solutions should agree with the direct asymptotic solution. Fortunately the solutions for two of the profiles shown in diagram 5 are already given in terms of Laguerre functions and it is easily demonstrated that as $k\to\infty$ the direct asymptotic technique gives the correct results. For the third profile it is quite a tedious and unfamiliar task to express the asymptotic form of the exact solution in terms of Laguerre function. A very simple though less rigorous test is by checking the asymptotic results for the wavelength. In general for situations with parallel depth contours it is found that

$$\left.\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}y} &= \left(\frac{\mathrm{d}h}{\mathrm{d}x}\bigg|_0\right)^{-1},\\ \frac{\mathrm{d}s_1}{\mathrm{d}y} &= -\frac{g^{-\frac{1}{2}}f}{2n+1}\left(\frac{\mathrm{d}h}{\mathrm{d}x}\bigg|_0\right)^{-1},\end{aligned}\right\} \tag{4.5.1}$$

etc.

For the third profile the results obtained from equations (4.5.1) and (4.5.2) agree with the asymptotic expansion of the exact solution.

The surprising way that the Coriolis force only affects terms of order $k^{-3}$ or less in equations (4.3.3) to (4.3.6) would in itself suggest that a regular perturbation scheme, such as the alternative asymptotic solution for edge waves, could be extended to allow for the Coriolis terms. However the mild-looking requirement that all the coefficients are non-singular on $S$ introduces terms of order $k^{-1}$ which are proportional to $f$ and consequently the alternative asymptotic solution is inapplicable. This feature, as noted by Reid (1958), severely limits the usefulness of any theoretical results for edge waves in which Coriolis terms are neglected.

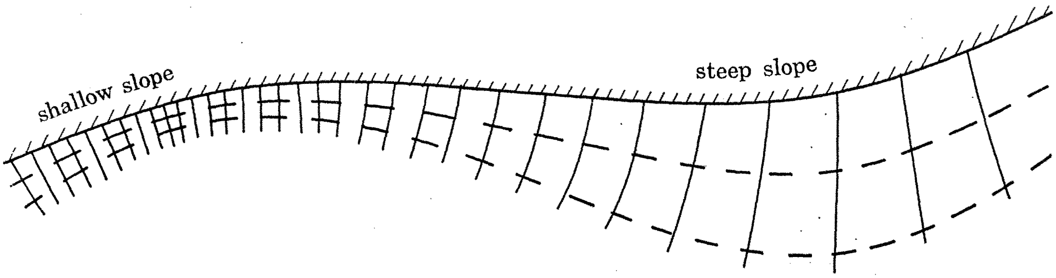


FIGURE 6. – – –, $r$ = constant; ——, $s$ = constant.

We can interpret the $r$ lines as indicating the approximate width of the high-energy zone and the $s$ lines as indicating approximate wave fronts. In figure 6 or from equations (4.3.13) to (4.3.17) we observe that adjacent to a shallow beach the $r$ and $s$ lines are closely packed together showing that the energy is trapped very close to the shore and is propagating very slowly along the shore. Each one of these effects tends to increase the height of the waves on the shore and from energy considerations we can deduce that the wave height on the shore is roughly inversely proportional to the beach slope. Detailed analysis shows that although $A_0$ is given simply by this result further terms have more complicated behaviour along $S$.

## 5. Surface waves

### 5.1. *Introduction*

Surface waves are disturbances which propagate along surfaces and whose amplitudes decay exponentially away from the surface. They are found in most situations where wave propagation takes place being easily distinguished from other classes of waves by their lower velocity of propagation. In elasticity, for example, Rayleigh waves propagate along the free surface of an elastic material and play a major part in the theory of seismology. Again in electromagnetism surface waves are used in some wave guides being propagated along corrugated metal or dielectric coated surfaces (Barlow & Brown 1962).

Of all the classes of wave motions studied in this paper surface waves are by far the most varied. Out of these many examples we shall follow Grimshaw (1968) and restrict attention to the representative problem of scalar waves $u(\boldsymbol{x})$ which satisfy the reduced wave equation

$$\nabla^2 u + k^2 N^2 u = 0 \tag{5.1.1}$$

and are subject to the linear homogeneous boundary condition

$$\partial u/\partial n + kYu = 0 \tag{5.1.2}$$

on the surface $S$. Where $N$ is the refractive index, $\partial/\partial n$ denotes differentiation along the inward normal $\boldsymbol{n}$ to the surface $S$ and $Y$ is a complex function of position on $S$ such that

$$Y = Y_R + \mathrm{i}Y_I \tag{5.1.3}$$

and

$$Y_R > 0, \quad Y_I \geqslant 0. \tag{5.1.4}$$

### 5.2. *Fundamental solution for surface waves*

As the frequency increases the effective width of a surface wave shrinks as $k^{-1}$. Applying the argument given in § 2.2 we deduce that in order to find the 'fundamental solution' for surface waves we need only study problems in which $S$ is perfectly flat, the refractive index $N$ is a function only of the perpendicular distance $x$ from the boundary and $Y$ is a constant.

We seek solutions of equations (5.1.1) and (5.1.2) corresponding to surface waves propagating along $S$

$$u = \hat{u}(x)\exp(\mathrm{i}k\hat{l}y), \tag{5.2.1}$$

where the real part of $l$ is positive. By substituting this representation for $u$ into the equations (5.1.1) and (5.1.2) together with the trapped wave requirement that $u$ tends to zero away from $S$, we get an eigenvalue problem for $\hat{u}$ and $\hat{l}$:

$$\mathrm{d}^2\hat{u}/\mathrm{d}x^2 + k^2(N^2 - l^2)\,\hat{u} = 0, \tag{5.2.2}$$

$$\text{on} \quad x = 0 \quad \mathrm{d}\hat{u}/\mathrm{d}x + kY\hat{u} = 0, \tag{5.2.3}$$

$$\text{as} \quad x \to \infty \quad \hat{u} \to 0. \tag{5.2.4}$$

The local form of the solutions near $x = 0$ as $k \to \infty$ can easily be obtained by making the change of independent variable

$$\xi = kx. \tag{5.2.5}$$

If now we let $k \to \infty$ we obtain a particularly simple eigenvalue problem for $\hat{u}_\infty$ and $\hat{l}_\infty$ with the solutions

$$\hat{u}_\infty(\xi) = C\exp(-Y\xi), \tag{5.2.6}$$

and

$$\hat{l}_\infty = (N^2(0) + Y^2)^{\frac{1}{2}}, \tag{5.2.7}$$

in which $C$ is an arbitrary constant.

This local solution defines the 'fundamental solution' for surface waves but since $Y$ is complex the $r$ and $s$ directions do not coincide with the $x$ and $y$ directions. In the notation of § 1.2

$$f_1(t) = \exp(-t) \tag{5.2.8}$$

and $f_1$ satisfies the ordinary differential equation

$$\mathrm{d}^2 f_1/\mathrm{d}t^2 - f_1 = 0, \tag{5.2.9}$$

therefore

$$f_2(t) = -\exp(-t). \tag{5.2.10}$$

### 5.3. *Derivation of equations*

We now return to the general situation and seek a solution for surface waves propagating along a smooth curved surface $S$ bounding a non-homogeneous medium and on which $Y$ is a function of position. We use the 'fundamental solution' to obtain the general solution for surface waves

$$u = (A - \mathrm{i}B)\exp(\mathrm{i}k\sigma - k\rho), \tag{5.3.1}$$

where $A$, $B$, $\sigma$ and $\rho$ are all real slowly varying functions of position which may depend on $k$. The $k^0$ factor multiplying $B$ is found from the requirement that when the wave equation is used the $A$ and $B$ terms are of the same order in $k$.

By substituting the general solution (5.3.1) directly into the reduced wave equations (5.1.1) then separating real and imaginary parts we obtain

$$k^2\{A((\nabla\rho)^2 - (\nabla\sigma)^2 + N^2) - 2B\nabla\rho.\nabla\sigma\} + k\{-\nabla.(A\nabla\rho) - \nabla\rho.\nabla A + \nabla.(B\nabla\sigma) + \nabla\sigma.\nabla B\} + \nabla^2 A = 0 \tag{5.3.2}$$

and

$$k^2\{-B((\nabla\rho)^2 - (\nabla\sigma)^2 + N^2) - 2A\nabla\rho.\nabla\sigma\} + k\{\nabla.(A\nabla\sigma) + \nabla\sigma.\nabla A + \nabla.(B\nabla\rho) + \nabla\rho.\nabla B\} - \nabla^2 B = 0. \tag{5.3.3}$$

We now have two equations in four unknowns instead of the usual four equations. This peculiarity is due to the two functions $f_1$ and $f_2$ no longer being independent. However, if we formally treat $f_1$ and $f_2$ as being different functions we find that the $k$ coefficients in the above two equations must vanish. The resulting four equations are most conveniently expressed as

$$(\nabla\rho)^2 - (\nabla\sigma)^2 + N^2 + \frac{1}{k^2}\frac{A\nabla^2 A + B\nabla^2 B}{A^2 + B^2} = 0, \tag{5.3.4}$$

$$\nabla\rho.\nabla\sigma + \frac{1}{k^2}\frac{A\nabla^2 B - B\nabla^2 A}{A^2 + B^2} = 0, \tag{5.3.5}$$

$$\nabla.(A\nabla\sigma) + \nabla\sigma.\nabla A + \nabla.(B\nabla\rho) + \nabla\rho.\nabla B = 0, \tag{5.3.6}$$

and

$$\nabla.(A\nabla\rho) + \nabla\rho.\nabla A - \nabla.(B\nabla\sigma) - \nabla\sigma.\nabla B = 0. \tag{5.3.7}$$

Any complex multiple of $u$ is also a surface wave, thus there is no distinction between $A$ and $B$ in the equation (5.3.1). We adjust this unsatisfactory state of affairs by requiring that near $S$ the amplitude correction term $B$ should be as small as possible without violating the equations or boundary conditions. This requirement gives two boundary conditions

$$\text{on } S \quad B = 0, \tag{5.3.8}$$

and

$$\text{on } S \quad \partial B/\partial n = 0. \tag{5.3.9}$$

The original boundary condition (5.1.2) gives us two further boundary conditions namely

$$\text{on } S \quad \partial\sigma/\partial n + Y_I = 0 \tag{5.3.10}$$

and

$$\text{on } S \quad Y_R - \frac{\partial\rho}{\partial n} + \frac{1}{k}\frac{1}{A}\frac{\partial A}{\partial n} = 0. \tag{5.3.11}$$





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From the four equations (5.3.4) to (5.3.7) and the four boundary conditions (5.3.8) to (5.3.11) we deduce that as $k \to \infty$ $A$, $B$, $\rho$ and $\sigma$ must possess asymptotic expansions of the forms

$$A \sim A_0 + A_1/k + \ldots, \tag{5.3.12}$$

$$B \sim B_0 + B_1/k + \ldots, \tag{5.3.13}$$

$$\rho \sim r + r_1/k + \ldots \tag{5.3.14}$$

and

$$\sigma \sim s + s_1/k + \ldots, \tag{5.3.15}$$

in which $A_j$, $B_j$, $r_j$ and $s_j$ are all independent of $k$.

By substituting these expansions back into the equations and boundary conditions and extracting like-powers of $k$ we obtain the following sequence of equations:

$$(\nabla r)^2 - (\nabla s)^2 + N^2 = 0, \tag{5.3.16}$$

$$\nabla r . \nabla s = 0, \tag{5.3.17}$$

$$\text{on} \quad S \quad \partial r/\partial n = Y_R, \tag{5.3.18}$$

$$\text{on} \quad S \quad \partial\sigma/\partial n = -Y_I, \tag{5.3.19}$$

$$\nabla . (A_0 \nabla s) + \nabla s . \nabla A_0 + \nabla . (B_0 \nabla r) + \nabla r . \nabla B_0 = 0, \tag{5.3.20}$$

$$\nabla . (A_0 \nabla r) + \nabla r . \nabla A_0 - \nabla . (B_0 \nabla s) - \nabla s . \nabla B_0 = 0, \tag{5.3.21}$$

$$\text{on} \quad S \quad B_0 = 0, \tag{5.3.22}$$

$$\text{on} \quad S \quad \partial B_0/\partial n = 0, \tag{5.3.23}$$

$$\nabla r . \nabla r_1 - \nabla s . \nabla s_1 = 0, \tag{5.3.24}$$

$$\nabla r . \nabla s_1 + \nabla s . \nabla r_1 = 0, \tag{5.3.25}$$

$$\text{on} \quad S \quad \frac{\partial r_1}{\partial n} = \frac{1}{A_0} \frac{\partial A_0}{\partial n}, \tag{5.3.26}$$

$$\text{on} \quad S \quad \partial s_1/\partial n = 0, \tag{5.3.27}$$

etc.

Further equations closely resemble equations (5.3.20) to (5.3.27) with progressively larger non-homogeneous terms on the right-hand sides.

From these equations once initial conditions are given we can solve successively for $r$, $s$; $A_0$, $B_0$; $r_1$, $s_1$; etc. The form of the equations indicate that in real situations numerical calculations would be quite straightforward, each group of four equations consisting of two well behaved first degree equations and two boundary conditions in two unknowns.

### 5.4. *Discussion of results*

The $s$ and $r$ coordinate curves, as shown in figure 7, give a simple picture of the shapes of wavefronts and lines of equal amplitude when the frequency is high. Two necessary conditions for a wave to be a surface wave are that the amplitude decreases exponentially outwards from the surface and that the amplitude does not increase exponentially in the direction of propagation. Using these conditions for very high-frequency waves propagating along $s$ increasing we deduce that

$$\text{on} \quad S \quad \partial r/\partial n > 0 \quad \text{and} \quad \partial s/\partial n \leqslant 0. \tag{5.4.1}$$

This shows the origin of the assumptions (5.1.4) on the values of $Y_R$ and $Y_I$. As $k \to \infty$ these conditions become sufficient for the existence of surface waves independently of any properties of the medium which explains the relative predominance of surface waves as compared to other trapped wave motions.

It is essential to check whether, in those ideal problems for which exact analytic solutions are known, the asymptotic expansions of the exact solutions agree with the direct asymptotic solutions. For problems in which the refractive index has a discontinuity away from $S$ the direct method seems to give all the terms which are algebraic in $k$ but fails to detect any terms exponential in $-k$, consequently in such problems the direct asymptotic solution is only useful when the algebraic terms exceed the ignored exponential terms or equivalently when the wavelength of the surface wave is much less than the distance of the discontinuity from $S$. For problems in which the refractive index is analytic all known asymptotic expansions of the exact solutions do not include any terms exponential in $-k$, as this observation suggests, I have never found any disagreement between the asymptotic expansions of the exact solutions and the direct asymptotic solution. An example of such a comparison is given below.

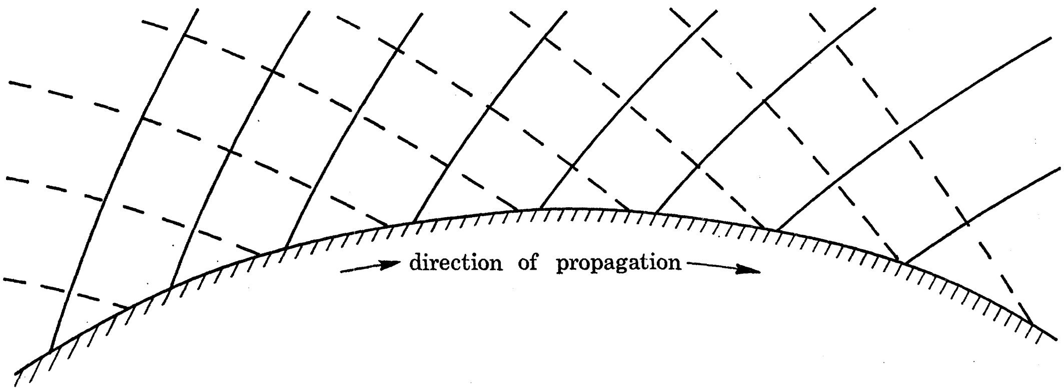


FIGURE 7. – – –, $r$ = constant; ——, $s$ = constant.

We study the problem of axially symmetric surface waves propagating along a circular cylinder which is surrounded by a homogeneous medium with $N \equiv 1$ and on the boundary $Y$ is a real constant. The exact solution is

$$u = \exp\left(\mathrm{i}k(\gamma^2+1)^{\frac{1}{2}} z\right) \frac{K_0(k\gamma R)}{K_0(k\gamma)}, \tag{5.4.2}$$

where $K_0$ is a modified Bessel function of order zero and $\gamma$ is determined from the boundary condition (5.1.2) on $R = 1$

$$K_0'(k\gamma) + Y.K_0(k\gamma) = 0. \tag{5.4.2}$$

From the simple geometry and boundary conditions it can easily be shown that $B \equiv 0$, $A$ and $\rho$ are independent of the distance along the cylinder, i.e. the $z$ direction while $\sigma$ is independent of $R$. Consequently equations (5.3.16) to (5.3.27) take the simplified forms

$$\left(\frac{\mathrm{d}r}{\mathrm{d}R}\right)^2 - \left(\frac{\mathrm{d}s}{\mathrm{d}z}\right)^2 + 1 = 0, \tag{5.4.3}$$

$$\text{on } R = 1 \quad \mathrm{d}r/\mathrm{d}R = Y, \tag{5.4.4}$$

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}\left(RA_0\frac{\mathrm{d}r}{\mathrm{d}R}\right) + \frac{\mathrm{d}A_0}{\mathrm{d}R}\frac{\mathrm{d}r}{\mathrm{d}R} = 0, \tag{5.4.5}$$

$$\frac{\mathrm{d}r}{\mathrm{d}R}\frac{\mathrm{d}r_1}{\mathrm{d}R}-\frac{\mathrm{d}s}{\mathrm{d}z}\frac{\mathrm{d}s_1}{\mathrm{d}z}=0 \tag{5.4.6}$$

and on $R=1$
$$\frac{\mathrm{d}r_1}{\mathrm{d}R}=\frac{1}{A_0}\frac{\mathrm{d}A_0}{\mathrm{d}r}. \tag{5.4.7}$$

We can easily solve these equations to obtain

$$s=(1+Y^2)^{\frac{1}{2}}z, \tag{5.4.8}$$

$$r=Y(R-1), \tag{5.4.9}$$

$$A_0=R^{-\frac{1}{2}}, \tag{5.4.10}$$

$$s_1=-\tfrac{1}{2}Y(1+Y^2)^{-\frac{1}{2}}z \tag{5.4.11}$$

and
$$r_1=\tfrac{1}{2}(R-1). \tag{5.4.12}$$

To find the leading terms in the asymptotic expansion of the exact solution we use the expansion

$$K_0(t)\sim(\pi/2t)^{\frac{1}{2}}\exp(-t), \tag{5.4.13}$$

which is valid for large $t$. The boundary condition (5.4.2) gives

$$\gamma\sim Y-1/2k. \tag{5.4.14}$$

Substituting this value for $\gamma$ into the exact solution (5.4.2) and expanding the Bessel functions by means of equation (5.4.13) we find that the exact solution agrees with the direct asymptotic solution in the leading terms. The next approximation has also been checked but demands much more effort than does the above analysis.

### 5.5. *Relations to other methods*

The problem which Grimshaw (1968) solves is identical with the one treated in this section. He regarded geometrical optics with complex rays as giving a uniformly valid solution for large $k$ and supplied the correction terms via the expansion

$$u\sim\exp(\mathrm{i}k\phi)\sum_{m=0}^{\infty}\bar{A}_n(\mathrm{i}k)^{-m}, \tag{5.5.1}$$

in which $\phi$ and $\bar{A}_n$ are independent of $k$. This solution is equivalent to demanding that in the general solution (5.3.1) $\rho$ and $\sigma$ are independent of $k$ while $A$ and $-\mathrm{i}B$ are combined into one complex term $\bar{A}$.

The relations between the two alternative expansions can be found by expanding by the use of Taylor's series the results of the new asymptotic solution when $k$ is very large. This gives

$$\phi=s+\mathrm{i}r, \tag{5.5.2}$$

$$\bar{A}_0=(A_0-\mathrm{i}B_0)\exp(-r_1+\mathrm{i}s_1), \tag{5.5.3}$$

etc.

In one sense Grimshaw's expansion is the more efficient since in order to obtain a uniformly valid solution for $u$ in a finite region as $k\to\infty$ we need only find $\bar{A}_0$ and $\phi$ while the new expansion requires $A_0$, $B_0$, $r$, $s$, $r_1$ and $s_1$. As one of the examples of his expansion Grimshaw solves the problem of surface waves propagating along a circular cylinder surrounded by a homogeneous medium and on which $Y$ is a real constant. On the other hand, not only does the solution fail to detect the obvious symmetry of the problem but it also camouflages the dispersion, in complete contrast to the treatment of that problem given in § 5.4. Although such features might be tolerated at high frequencies; at lower frequencies the new solution is much more desirable.

Since the new solution is closely related to Grimshaw's solution which is a ray solution, we should expect that it is possible to describe $A_0$, $B_0$; $r_1$, $s_1$, etc., using rays. This conclusion is supported by the changes of variables

$$\phi^{\pm} = s \pm \mathrm{i}r, \tag{5.5.4}$$

$$z_0^{\pm} = A_0 \mp \mathrm{i}B_0, \tag{5.5.5}$$

$$\phi_1^{\pm} = s_1 \pm \mathrm{i}r_1, \tag{5.5.6}$$

etc.,

which lead to the sequence of equations

$$(\nabla\phi^{\pm})^2 = N^2, \tag{5.5.7}$$

$$2\nabla\phi^{\pm}.\nabla z_0^{\pm} + z_0^{\pm}\nabla^2\phi^{\pm} = 0, \tag{5.5.8}$$

$$\nabla\phi^{\pm}.\nabla\phi_0^{\pm} = 0, \tag{5.5.9}$$

etc.

Equations (5.5.7) and (5.5.8) can be recognized as being the 'eikonal' and 'transport' equations respectively of geometrical optics. The most noticeable difference between this ray interpretation and Grimshaw's is that he uses only one family of complex rays $\phi^+$ not two.

As observed for other ray interpretations of trapped wave problems, although the above equations become ordinary differential equations along the complex rays and can be solved by marching along these complex directions it is usually easier to solve the previous set of equations (5.3.16) to (5.3.27).

There are many further ways of expressing the asymptotic solution for surface waves, one particularily neat form being

$$u = \exp(\mathrm{i}kp - kq), \tag{5.5.10}$$

where $p$ and $q$ are related to $A$, $B$, $\rho$ and $\sigma$ by the two equations

$$p = \sigma - k^{-1}\tan^{-1}(B/A) \tag{5.5.11}$$

and

$$q = \rho - \tfrac{1}{2}k^{-1}\ln(A^2+B^2). \tag{5.5.12}$$

However, the equations derived in § 5.3 are in the most natural form in the light of the equations derived for other classes of waves where such alternative forms of the general solution are not always possible.

## 6. Limitations of the technique

The asymptotic solution is essentially a boundary-layer solution based upon the assumptions that within the layer all properties vary slowly and that the behaviour of the wave in that layer is insensitive to any exterior complications. Consequently the limitations of the technique correspond to the limitations upon the width of the boundary layer.

The presence of discontinuities in the medium or of other boundaries or coasts puts obvious limits upon the width of the boundary layer. If such places are several times the boundary layer width from $S$ the value of $u$ will be exponentially small there and correspondingly the correction to the value of $u$ within the boundary layer will be exponentially small.



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Far away from $S$ the coordinate surfaces can have various complications, for example as shown in figure 8 the surfaces can cross over giving multiple $(r, s)$ values to a single point. Provided that the surfaces do not focus perfectly we can, except near the innermost crossing curve labelled $P$, simply superimposed the various solutions. The solution with the smallest value of $r$ will of course dominate. The presence of a focus at $P$ will alter the value of $u$ within the boundary layer so we must ensure that the width of the boundary layer is much less than the distance out to $P$ in order that the change in $u$ is exponentially small.

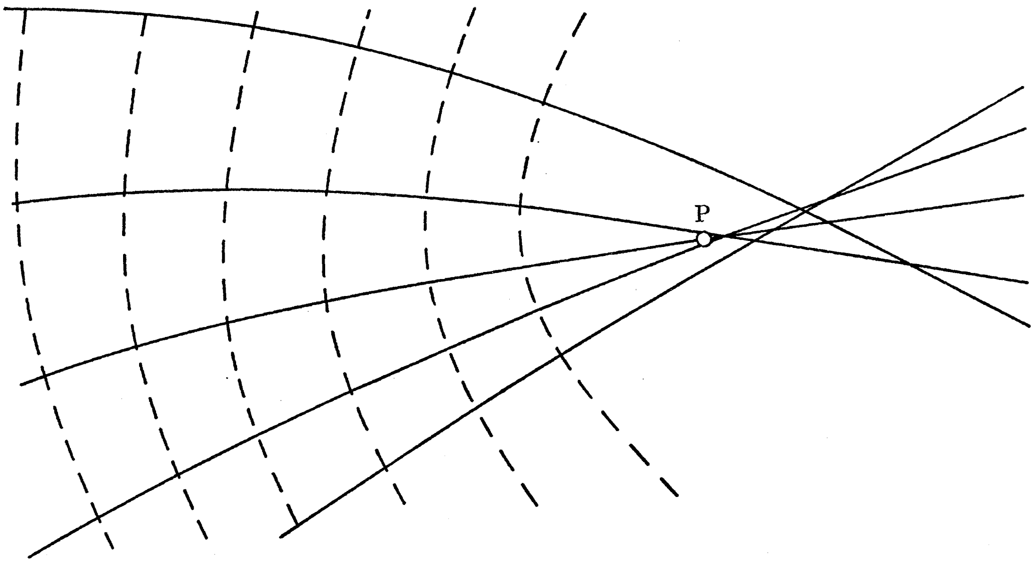


FIGURE 8. – – –, $r$ = constant; ——, $s$ = constant.

Another more crucial phenomenon sometimes found in the geometry of the $r$ and $s$ surfaces is that the construction of the surfaces fails far away for $S$ unless we are willing to allow $r$ and $s$ to becomes complex. The surfaces along which the construction breaks down are caustics to those complex rays associated with $k = \infty$. Beyond the caustics there are real rays which can either leak energy out of or receive energy into the trapped wave modes, in order to be consistent with our local view of wave trapping we can only study the leaky case. Because $C$ is far away from $S$ both $u$ and the rate of loss of wave energy will be exponentially small, however the effect upon the value of $u$ near $S$ is systematic and might be important in some problems; see, for example, Lord Rayleigh (1914) and Longuet-Higgins (1967).

It is inevitable that for a real situation the final equations will be solved numerically. The grid of calculation points will necessarily be quite widely spaced so the real problem is in effect imitated numerically by a considerably smoothed model. We can justify the use of a smooth model by the very plausible and in some circumstances rigorous principle that individual details of a problem do not matter so that results for the smoothed problem are very close to real results. Similarly, when we use asymptotic methods to solve the smoothed problem we know that when $k$ is large the asymptotic results are close to the results for the smoothed problem and hence close to the real results.

As computers become faster it will become possible to use a fine grid of calculation points which if we wished could give a closer model to the real situation. However, if we used a more jagged model we might find the asymptotic technique's predictions became less accurate at low frequencies. For example, in the study of long waves trapped over an oceanic ridge the particular details of the highest peak on the ridge would become important even though waves short enough to be trapped over this one peak might not be long waves. Consequently when solutions are found by the asymptotic technique the model giving the best solutions is smooth and may depend weakly upon $k$, while the scattering and other effects due to the jaggedness of the real situation must be found by some additional technique.

It should be noted that the technique has been developed only for scalar waves. The appendix to this paper shows one way in which the technique can be generalized for a system of first-order equations. Unfortunately the final sequence of equations is no longer simple so the generalization may only be of limited interest.

## 7. Conclusion

From the many applications to different types of trapped wave problems and the asymptotic agreement with exact analytic solutions it is evident that the technique developed in this paper provides a natural and efficient means of analysing high-frequency trapped scalar waves. It also appears that for real situations the sequence of equations obtained by using this technique are very amenable to numerical solutions.

I wish to thank Dr Howell Peregrine for many helpful discussions and the referees for their constructive comments. This work was financed by an S.R.C. research-studentship and is to form part of my Ph.D. thesis for the University of Bristol.

## 8. Appendix

### 8.1. *Asymptotic solutions of a first-order system*

Until now the asymptotic technique has been developed exclusively for scalar waves. In some circumstances we cannot reduce the equations governing wave propagation to a single scalar equation, so in this appendix we shall generalize the asymptotic technique in order to find asymptotic solutions of a first-order system:

$$\sum_{j=0}^{n} L^j \cdot \frac{\partial u}{\partial x^j} + M . u = 0. \tag{A 1}$$

Here $\boldsymbol{x} = (x_0, \dots, x_n)$ represent the independent variables, with $x_0$ usually thought of as time. The coefficients $L^j$ and $M$ are square matrices which are assumed to be both real and smooth, and $u(\boldsymbol{x})$ is a vector.

To avoid the complication of explicit boundary conditions we shall only study the propagation of ducted waves, the extension to other classes of trapped waves is straightforward. The general solution no longer takes the form (3.3.1) but the slightly more complicated form

$$u = \exp(\mathrm{i}k\sigma)\,\{a\Psi_n(k^{\frac{1}{2}}\rho) + \mathrm{i}k^{-\frac{1}{2}}b\Psi_n'(k^{\frac{1}{2}}\rho)\} + \mathrm{i}/k\exp(\mathrm{i}k\sigma)\,\{c\Psi_n(k^{\frac{1}{2}}\rho) + \mathrm{i}k^{-\frac{1}{2}}\,d\Psi_n'(k^{\frac{1}{2}}\rho)\}, \tag{A 2}$$

in which $a, b, c, d, \rho, \sigma$ are all real slowly varying functions of position. The only modification to the previous general solution (3.3.1) is the inclusion of the additional correction terms $c$ and $d$.

By substituting the general solution (A 2) into equation (A 1) and extracting like terms we obtain the following four vector equations:

$$L^j \frac{\partial \sigma}{\partial x^j} . a + L^j \frac{\partial \rho}{\partial x^j} . \rho^2 b - \frac{2n+1}{k} L^j \frac{\partial \rho}{\partial x^j} . b + \frac{1}{k^2}\left(L^j \frac{\partial c}{\partial x^j} + M . c\right) = 0, \tag{A 3}$$

$$L^j\frac{\partial\rho}{\partial x^j}.a - L^j\frac{\partial\sigma}{\partial x^j}.b - \frac{1}{k^2}\left(L^j.\frac{\partial d}{\partial x^j} + M.c\right) = 0, \tag{A 4}$$

$$L^j\frac{\partial\sigma}{\partial x^j}.c + L^j\frac{\partial\rho}{\partial x^j}\rho^2.d - L^j.\frac{\partial a}{\partial x^j} - M.a - \frac{2n+1}{k}L^j\frac{\partial\rho}{\partial x^j}.d = 0, \tag{A 5}$$

$$L^j\frac{\partial\rho}{\partial x^j}.c - L^j\frac{\partial\sigma}{\partial x^j}.d + L^j.\frac{\partial b}{\partial x^j} + M.b = 0. \tag{A 6}$$

From these four vector equations we can deduce that as $k\to\infty$ $a, b, c, d, \rho, \sigma$ all have asymptotic expansions of the form

$$a \sim a_0 + a_1/k + \dots \text{ etc.} \tag{A 7}$$

By substituting these expansions back into the equations (A 3) to (A 6) and then extracting like-powers of $k$ we obtain the following sequence of equations:

$$L^j\frac{\partial s}{\partial x^j}.a_0 + L^j\frac{\partial r}{\partial x^j}r^2.b_0 = 0, \tag{A 8}$$

$$L^j\frac{\partial r}{\partial x^j}.a_0 - L^j\frac{\partial s}{\partial x^j}.b_0 = 0, \tag{A 9}$$

$$L^j\frac{\partial s}{\partial x^j}.c_0 + L^j\frac{\partial r}{\partial x^j}r^2.d_0 - L^j.\frac{\partial a_0}{\partial x^j} - M.a_0 = 0, \tag{A 10}$$

$$L^j\frac{\partial r}{\partial x^j}.c_0 - L^j\frac{\partial s}{\partial x^j}.d_0 + L^j.\frac{\partial b}{\partial x^j} + M.b_0 = 0, \tag{A 11}$$

$$L^j\frac{\partial s}{\partial x^j}.a_1 + L^j\frac{\partial r}{\partial x^j}r^2.b_1 + L^j\frac{\partial s_1}{\partial x^j}.a_0 + L^j\frac{\partial}{\partial x^j}(r^2r_1).b_0 = (2n+1)L^j\frac{\partial r_1}{\partial x^j}.b_0, \tag{A 12}$$

$$L^j\frac{\partial r}{\partial x^j}.a_1 - L^j\frac{\partial s}{\partial x^j}.b_1 + L^j\frac{\partial r_1}{\partial x^j}.a_0 - L^j\frac{\partial s_1}{\partial x^j}.b_0 = 0, \tag{A 13}$$

etc.

Further equations resemble equations (A 10) to (A 13) with progressively larger non-homogeneous right-hand sides.

These equations are considerably more complicated than the corresponding sequence of equations (3.3.11) to (3.3.16) obtained for scalar ducted waves, being both vector equations and under-determined. Consequently we shall demonstrate one procedure by which the above equations can be solved in principle.

We can combine the first two equations (A 8) and (A 9) into the two relationships

$$\left(L^j\frac{\partial s}{\partial x^j} \pm \mathrm{i}L^j\frac{\partial r}{\partial x^j}r\right)(a_0 \mp \mathrm{i}rb_0) = 0. \tag{A 14}$$

Thus we must have

$$\det\left(L^j\frac{\partial s}{\partial x^j} \pm \mathrm{i}L^j\frac{\partial r}{\partial x^j}r\right) = 0. \tag{A 15}$$

For simplicity we shall assume that these two equations uniquely determine $r$ and $s$. It follows from these two scalar equations that there exist pairs of vectors $e_1$, $e_2$ and $f_1, f_2$ such that

$$e_1.L^j\frac{\partial s}{\partial x^j} + e_2.L^j\frac{\partial r}{\partial x^j}r^2 = 0, \tag{A 16}$$

$$e_1.L^j\frac{\partial r}{\partial x^j} - e_2.L^j\frac{\partial s}{\partial x^j} = 0, \tag{A 17}$$

and
$$L^j \frac{\partial s}{\partial x^j}.f_1 + L^j \frac{\partial r}{\partial x^j} r^2.f_2 = 0, \tag{A 18}$$

$$L^j \frac{\partial r}{\partial x^j}.f_1 - L^j \frac{\partial s}{\partial x^j}.f_2 = 0. \tag{A 19}$$

From equations (A 8) and (A 9), together with equations (A 18) and (A 19) we can deduce that

$$a_0 = p_0 f_1 \tag{A 20}$$

and
$$b_0 = q_0 f_2, \tag{A 21}$$

where $p_0$ and $q_0$ are scalar functions which have yet to be determined. Fortunately the two relationships (A 16) and (A 17) enable us to combine equations (A 10) and (A 11) into two scalar equations which we use to obtain $p_0$ and $q_0$:

$$e_1.\left(L^j.\frac{\partial a_0}{\partial x^j} + M.a_0\right) - r^2 e_2.\left(L^j.\frac{\partial b_0}{\partial x^j} + M.b_0\right) = 0 \tag{A 22}$$

and
$$e_2.\left(L^j.\frac{\partial a_0}{\partial x^j} + M.a_0\right) + e_1.\left(L^j.\frac{\partial b_0}{\partial x^j} + M.b_0\right) = 0. \tag{A 23}$$

Once $a_0$ and $b_0$ have been found we can solve equations (A 10) and (A 11) for $c_0$ and $d_0$ except for arbitrary multiples of the vectors $f_1$ and $f_2$ respectively. Unfortunately the original sequence of equations is under-determined so we shall make the additional demands that $c_k$ and $d_k$ satisfy equations of the forms (A 22) and (A 23). These demands enable us to uniquely determine $c_k$ and $d_k$ and also appear to make the solutions obtained from a scalar wave equation identical with solutions obtained from the corresponding first-order system.

Using the relationships (A 16) and (A 17) we can combine equations (A 12) and (A 13) into two scalar equations which we use to obtain $r_1$ and $s_1$:

$$e_1.\left(L^j \frac{\partial s_1}{\partial x^j}.a_0 + L^j \frac{\partial}{\partial x^j}(r^2 r_1).b_0\right) + r^2 e_2.\left(L^j \frac{\partial r_1}{\partial x^j}.a_0 - L^j \frac{\partial s_1}{\partial x^j}.b_0\right) = e_1.(2n+1)\, L^j \frac{\partial r}{\partial x^j}.b_0 \tag{A 24}$$

and
$$e_2.\left(L^j \frac{\partial s_1}{\partial x^j}.a_0 + L^j \frac{\partial}{\partial x^j}(r^2 r_1).b_0\right) - e_1.\left(L^j \frac{\partial r_1}{\partial x^j}.a_0 - L^j \frac{\partial s_1}{\partial x^j}.b_0\right) = e_2.(2n+1)\, L^j \frac{\partial r}{\partial x^j}.b_0. \tag{A 25}$$

From this point onwards the sequence of operations repeats and further equations resemble those given above with progressively large non-homogeneous right-hand sides.

For a real situation obtaining solutions by means of the above procedure would be a formidable task. Regretably we must conclude that for problems which cannot be reduced to a single scalar wave equation the asymptotic technique may only provide a formal (as opposed to a practical) method for solving trapped wave problems.

## 9. References


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